SOME APPLICATIONS OF THE RAYLEIGH-RITZ METHOD TO THE THEORY OF THE STRUCTURE OF MATTER[1]

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I have been asked to discuss for you some applications of the Rayleigh-Ritz method in physics. Rather than attempt to survey the whole of a discouragingly large field I have preferred to restrict my attention to some few topics from the theory of atomic and molecular structure. The application of the Rayleigh-Ritz method in this field has been carried out with some care and persistence, and there are enough results available to make possible a critical discussion of the potentialities and limitations of the method. This field also presents good examples of how one's mathematical procedure is influenced by one's physical notions concerning the system to be studied, and, in turn, what sort of physical conclusions may be drawn from results yielded by the method.

First of all I shall indicate why recourse to the Ritz[2] method is necessary if one is to obtain a satisfactorily accurate treatment of some of the principal problems in atomic and molecular theory. I shall indicate, though not in as much detail as I should like, the nature of the considerations by which one chooses the set of coordinate functions employed in this method, and show how rapid a convergence one can hope for in the approach to the solution. After presentation of some of the very satisfactory results one can obtain, by the application of enough hard work, in not too unfavorable cases, I shall finally have to indicate the increasingly serious limitations of the method as one seeks to apply it to the treatment of more and more complicated physical systems.

The fundamental mathematical problem in the study of an atomic system by wave mechanics is usually the solution of the stationary state wave equation for that system. In the non-relativistic case this is a linear partial differential equation of the second order, which can be symbolized thus:

$$H\psi = E\psi.$$

Here H represents the Hamiltonian operator for the system obtained

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[1] An address delivered before the meeting of the Society in Washington, D. C., on May 3, 1941, by invitation of the Program Committee.

[2] W. Ritz, Journal für die reine und angewandte Mathematik, vol. 135 (1909), p. 1.

from the Hamiltonian function H of classical mechanics by the operator substitution

$$p_i \rightarrow \frac{h}{2\pi i} \frac{\partial}{\partial q_i}$$

where h is Planck's constant. For instance, for a hydrogen atom with fixed nucleus H is the sum of the kinetic energy of the one electron and the potential energy of the electron with respect to the nucleus:

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) - \frac{e^2}{r}.$$

The Hamiltonian operator correspondingly involves the Laplacian operator in a term representing the kinetic energy of the electron and a scalar multiplier representing the potential energy:

$$H = -\frac{h^2}{8\pi^2 m}\nabla^2 - \frac{e^2}{r}.$$

The wave equation is then

$$-\frac{h^2}{8\pi^2 m}\nabla^2\psi - \frac{e^2}{r}\psi = E\psi.$$

In these equations E is a number, the energy of the system, and ψ, the corresponding solution, is the wave function which describes the state of the system when it has that energy. ψ is obviously a function of all the spatial coordinates needed to define a configuration of the system. In the case of a simple molecule like Cl_2, for instance, ψ would depend on 108 variables. It is this multiplicity of variables which makes it so hard for physicists to interest mathematicians in some problems they would like very much to have solved.

Now not all the solutions of the wave equation are of interest, but only those which can be physically interpreted. This requirement of physical interpretability places certain auxiliary conditions on the functions ψ, often merely that they be quadratically integrable or that they remain bounded. Between the *physically interpretable* wave functions and the stationary states in which the system may be found there is a one-to-one correspondence. Thus in many cases physically satisfactory solutions of the wave equation exist only for certain discrete values of E, and in such cases one finds that the physical system also can possess only these discrete energies.

The physically satisfactory solutions ψ of such an equation and the

corresponding energies E form the raw material for any detailed quantitative treatment of the atomic system. For instance, a molecule can emit radiation when its energy changes from one of the discrete possible values to another. One can compute the average intensity of such radiation as soon as one knows the ψ functions for the initial and final stationary states, while the wavelength can be determined at once from the corresponding E's. Thus by determining all the physically acceptable ψ's and E's which satisfy the wave equation for a molecule one could make a complete prediction of the wavelengths and intensities in the spectrum emitted by a gas consisting of these molecules, under any specified conditions. These wave functions are similarly fundamental in the study of collisions and other interactions between atoms or molecules, the absorption and scattering of radiation, and so on. The difficulty is to get reasonably accurate ψ functions on which to base further work.

In the simplest case of the hydrogen atom a complete solution of the problem is possible. The wave equation is separable in spherical coordinates, and one has to deal only with ordinary differential equations which offer no special difficulty. However, when one passes to the next more complicated problem, the helium atom with a heavy nucleus and two electrons, one meets most of the difficulties characteristic of this work.

If we treat the nucleus of the atom as fixed, the wave equation includes terms representing the kinetic energies of the electrons, the potential energies of the first and second electrons with respect to the nucleus, and their potential energy with respect to each other:

$$-\frac{h^2}{8\pi^2 m}(\nabla_1^2+\nabla_2^2)\psi(r_1, r_2)+\left(-\frac{2e^2}{r_1}-\frac{2e^2}{r_2}+\frac{e^2}{r_{12}}\right)\psi(r_1, r_2)=E\psi(r_1, r_2).$$

It involves six independent variables or two vectors r_1, r_2, defining the positions of the two electrons; one might, for instance, take as three of these variables the separations r_1, r_2, r_{12} of the three particles, on which the potential energy terms depend, and as the other three variables the angles necessary to define the orientation in space of the triangle formed by the three particles.

The fundamental difficulty of this equation arises from the fact that it is not completely separable. One can separate out the angle variables just mentioned, but there remains a partial differential equation in three variables which is not separable in any set of coordinates. In the very simplest case, in which ψ does not depend on the angle variables at all, the separated equation becomes

$$-\frac{h^2}{8\pi^2 m}\left(\frac{\partial^2}{\partial r_1^2}+\frac{\partial^2}{\partial r_2^2}+2\frac{\partial^2}{\partial r_{12}^2}+\frac{2}{r_1}\frac{\partial}{\partial r_1}+\frac{2}{r_2}\frac{\partial}{\partial r_2}+\frac{4}{r_{12}}\frac{\partial}{\partial r_{12}}\right.$$
$$\left.+\frac{r_1^2-r_2^2+r_{12}^2}{r_1 r_{12}}\frac{\partial^2}{\partial r_1 \partial r_{12}}+\frac{r_2^2-r_1^2+r_{12}^2}{r_2 r_{12}}\frac{\partial^2}{\partial r_2 \partial r_{12}}\right)\psi$$
$$+\left(-\frac{2e^2}{r_1}-\frac{2e^2}{r_2}+\frac{e^2}{r_1}\right)\psi = E\psi.$$

This equation is not only complicated, but unpleasantly singular, the potential energy becoming infinite as any one of the three variables approaches zero. Though it does not seem outside the range of possibility, no one has yet been able to set up a direct method for solving this equation or even for proving the existence of solutions. One must immediately resort to approximations.

One method by which physicists often deal with such equations is that of forcing a separation of variables by neglect of judiciously chosen small terms, the resulting ordinary differential equations then being treated by the customary methods. This and the closely related methods of Hartree[3] and Fock[4] actually serve to give useful approximate solutions of atomic wave equations involving several hundreds of variables. Peculiarly enough, the limitations of these methods are in practice apt to be more serious the lighter, the less complicated, the atom.

In the case of the state of helium with lowest energy, the ground state, computations by this method become quite simple. If one makes the very crude approximation of neglecting the mutual repulsion of the electrons, as represented by the term e^2/r_{12}, the approximate wave function becomes

$$\psi = Ce^{-2(r_1+r_2)},$$

the unit of length being the atomic unit $a_H = 5.28\times10^{-9}$ cm. The atomic energy computed with this function is -76.43 electron volts, against an observed value of -80.62 electron volts. Neglect of a group of terms,

$$\frac{e^2}{r_{12}}-\frac{5}{32}\left(\frac{e^2}{r_1}+\frac{e^2}{r_2}\right),$$

[3] D. R. Hartree, Proceedings of the Cambridge Philosophical Society, vol. 24 (1928), p. 189; J. C. Slater, Physical Review, vol. 32 (1928), p. 339. Results published in many papers of Hartree, mostly in Proceedings of the Royal Society of London, series A.

[4] V. Fock, Zeitschrift für Physik, vol. 61 (1930), p. 126.

which is on the average smaller than the single term e^2/r_{12}, leads to a better wave function, $\psi = Ce^{-(27/16)(r_1+r_2)}$, and a computed atomic energy in error by 1.53 electron volts. Further improvements are not easily obtained, and by pressing this method to its limit one can reduce the error in the computed energy only to 0.81 electron volts.

To obtain wave functions and energy values more accurate than these, as one very much wants to do, one must resort to the variational formulation of the problem. One then seeks to determine the wave functions, not as solutions of a differential equation, but as the properly continuous functions which extremalize the average energy

$$\overline{E} = \frac{\int \psi^* H \psi d\tau}{\int \psi^* \psi d\tau};$$

the corresponding extremal values of this quantity are then the stationary state energies of the system, which one wants to determine.

A variety of variational procedures have been applied in the theory of atomic structure, particularly in the study of the lowest energy states of the systems. The correct wave function for such a state makes the computed energy an absolute minimum. An approximate wave function will then presumably be better the lower—that is, the more nearly correct—the corresponding energy. It is accordingly a direct, and very common, procedure to set up approximate wave functions known to have a generally appropriate form, but involving adjustable parameters $\alpha, \beta, \cdots$. Adjusting these parameters to minimize the average energy, one obtains the best approximate wave function of the assumed form, and possesses in the minimum value of $\overline{E}$ the most favorable upper limit on the ground state energy of the system.

For instance, one might assume a function of the form given by a forced separation of variables, but treat the constant in the exponential, which is the effective charge of the nucleus, as a variable parameter:

$$\psi(\alpha; \mathbf{r}_1, \mathbf{r}_2) = Ce^{-\alpha(r_1+r_2)}.$$

The best value of the parameter α is 27/16, the energy error being then 1.53 electron volts. Or one may set up the more elaborate and more flexible function,

$$\psi(\alpha, \beta; \mathbf{r}_1, \mathbf{r}_2) = C(e^{-\alpha r_1}e^{-\beta r_2} + e^{-\beta r_1}e^{-\alpha r_2}).$$

The best values of the parameters are then $\alpha = 2.15$, $\beta = 1.19$, and the error in the computed energy is only 0.76 electron volts, less than that given by any forced separation of variables.

Such schemes for the variation of parameters are often useful, but they have one serious limitation. In general an excessive amount of labor is required to determine the best values of the parameters, particularly if these occur in exponentials. As a result, the number of parameters which can be handled, and thus the flexibility of the trial functions, is seriously limited. This is one of the points at which the Rayleigh-Ritz method is superior to all others. Without excessive labor one can deal with Ritz functions containing as many as twenty parameters, whereas with other schemes one is in practice limited to two or three. The result is that the Ritz method is quite without a rival in the accurate computation of wave functions and atomic energies—as distinguished from roughly approximate computations.

The critical step in the application of the Ritz method is the choice of the set of coordinate functions in terms of which the wave functions are to be expressed as sums. Here one must balance against the need for reasonable ease in the computations the requirement of a general suitability in the character of the functions chosen, for even the considerable flexibility of the Ritz function in practice is not sufficient to compensate an inappropriate choice of the form of the component terms. One must bring to bear on this problem all the physical considerations which may be available. In the case of the ground state of helium the relatively simple arguments would run as follows:

First of all, since the electrons are identical the Hamiltonian operator is symmetrical to interchange of their coordinates. From this it follows easily that every He wave function should be either symmetric or antisymmetric to interchange of the coordinates of the electrons. States of the atom described by symmetric functions are those which the spectroscopist would class as singlet states, while the others are triplet states. The ground state of helium is known to be a singlet state; hence in attempting to determine its wave function one need consider only functions *symmetric to interchange of the electrons*. Similar general arguments,[5] combined with the spectroscopic observation that the ground state of helium is an S state, show that this wave function does not depend at all on the angle variables, but *depends only on the distances between the particles*.

There are other arguments of a less precise character. For instance, if either electron of the atom is very far away from the nucleus the chance is overwhelming that the second electron will be, relatively, very near to it. The electric field at the first electron would then be nearly the same as if the second electron and the nucleus were super-

[5] G. Breit, Physical Review, vol. 35 (1930), p. 569.

imposed—that is, almost Coulomb in character. Now we know that if the field were exactly Coulomb the wave function would decrease exponentially at large distances. Hence we can conclude that the situation will be similar in helium; the wave function should *decrease approximately exponentially* as the distance of either electron from the nucleus becomes very great.

This fact would be indicated by the approximate wave functions given by any of the methods previously discussed. But there is another important feature of the correct function which none of those approximate functions possesses; not one of them depends on the distance between the electrons. This is physically quite unreasonable, for it implies that the probability of any configuration of the system depends only on the electron-to-nucleus distances, and is just the same whether the electrons are very close together, on the same side of the nucleus, or far apart, on opposite sides of it. Obviously the mutual repulsion of the electrons will tend to keep them away from each other; correspondingly the wave function must tend to *decrease in magnitude with decreasing separation of the electrons.* It is the inability of most approximate wave functions to take account of this feature in the behavior of the system that is responsible for their largest errors.

How then shall one choose the terms from which the Ritz function is to be constructed? Formally, the simplest choice would be to make the Ritz function a power series, with the typical term,

$$r_1^m r_2^n r_{12}^p.$$

This is impossible, however, for no finite sum of such terms would show the desired exponential decrease for very large r_1 and r_2; the very integrals in terms of which the variational procedure is stated would not converge. But one can obtain the desired asymptotic character of the function, and assure the existence of all relevant integrals, by introducing an exponential factor into each term:

$$e^{-\delta(r_1+r_2)} r_1^m r_2^n r_{12}^p.$$

Introduction of an exponential factor in r_{12} might seem appropriate, but it would be quite unnecessary, and would only make the computations more tedious. A sum of such terms, in addition to the linear parameters characteristic of the Ritz method, would contain an exponential parameter for which it would be much more difficult to determine the very best value. Fortunately in practice one need worry little about the exact value assigned to such parameters; they may

be fixed once for all by an exploratory computation with a very few terms, or assigned a generally appropriate constant value on the basis of physical considerations of effective nuclear charges.

A finite sum of such terms can have all the properties one desires in respect to flexibility, asymptotic behavior and simplicity, but may lack the proper symmetry to interchange of the coordinates of the electrons. This symmetry one can obtain in various ways. For instance, one may construct the terms from powers of r_1+r_2 and r_1-r_2:

$$e^{-\delta(r_1+r_2)}(r_1+r_2)^m(r_1-r_2)^n r_{12}^p,$$

and assure the desired symmetry by including terms with all powers of (r_1+r_2), but only even powers of r_1-r_2. These were just the coordinate functions used by Hylleraas in his pioneer work on the theory of the helium atom.[6]

In his first work Hylleraas obtained a quite simple six-term function,

$$\psi = e^{-1.818(r_1+r_2)}\{c_0 + c_1 r_{12} + c_2 r_{12}^2 + c_3(r_1+r_2) + c_4(r_1+r_2)^2 + c_5(r_1-r_2)^2\}.$$

In the atomic units $R_{He}h$ the energy computed with this function is -5.80633, against the observed value of $-5.80736 \pm .00012$; the error of .0139 electron volts is about one-sixtieth that of the best previous computation. In later work[7] with an eight-term function the computed energy was depressed to -5.80748. It might seem embarrassing to the theory that the computed upper limit on the energy should come below the observed energy. However, the wave equation treated by Hylleraas involved certain approximations, neglecting relativistic effects and in part, the motion of the nucleus. Adding a correction of $+0.00010$ for the effect of nuclear motion and one of $+0.00018$ for relativistic effects[8] one obtains a theoretical value of -5.80720 for the ground state energy of helium. This agrees with the observed value to one part in 30,000, practically within the accuracy of observation, and, as was to be expected, lies slightly too high.

This very satisfactory treatment of the three-body problem in wave mechanics is of historical interest in that it gave the first quantitative check on the present theory of atomic structure as applied to systems more complicated than the hydrogen atom. Many theories have been devised to account for the very simple properties of hydro-

[6] E. A. Hylleraas, Zeitschrift für Physik, vol. 54 (1929), p. 347.
[7] E. A. Hylleraas, Zeitschrift für Physik, vol. 65 (1930), p. 209.
[8] H. Bethe, Handbuch der Physik, XXIV/1. See pages 371, 383.

gen, but no other one of them has been able to stand the test of extension to the treatment of helium. The confidence of the physicist in the essential correctness of the present atomic theory is of course based on a wide variety of comparisons with experiment. A number of other equally impressive examples of this particular quantitative type have been obtained through application of the Ritz method, and will be presented later.

By making use of the observed energies of the higher stationary states of helium one can fix an upper limit on the error of these wave functions.[9] One can show, for instance, that the root-mean-square error in the six-term function does not exceed 3 per cent; a reasonable estimate is 1 per cent. For the more complicated function the error is of the order of 1/5 per cent. Such accuracy is more than adequate for the purposes of the physicist.

Similar results can be obtained in studies of the lower excited states of helium. These states differ from the ground state in one important physical respect. In the ground state the two electrons are said to be equivalent; the most probable configurations of the atom are those in which the two electrons are at about the same distance from the nucleus. In the lower excited states of the atom just one of the electrons is excited; the most probable configurations are those in which one electron is much farther from the nucleus than the other. The effective nuclear charges are then different for the two electrons, that of the outer electron being reduced by its repulsion by the inner electron. This one should take into account in constructing the Ritz function, replacing

$$e^{-\delta(r_1+r_2)} r_1^m r_2^n r_{12}^p$$

by

$$e^{-\delta_1 r_1} e^{-\delta_2 r_2} r_1^m r_1^n r_{12}^p.$$

The terms in the Ritz function are then to be constructed by making appropriate linear combinations of these functions, symmetric in the case of ^1S terms, and antisymmetric in the case of ^3S terms.

If the excited state to be considered is not the lowest of its symmetry one does not obtain the energy of the state as the lowest root of a secular equation, but as the second lowest root, third lowest root, and so on. Fortunately one can show that an approximate energy thus computed is always an upper limit for the energy of the corresponding state of the atom, approaching the correct value from above as the

[9] H. M. James and A. S. Coolidge, Physical Review, vol. 51 (1937), p. 860.

flexibility and accuracy of the Ritz function are increased.[10,11] Consequently the error in the computed energy provides an indication (which can be made quantitative) of the accuracy of the corresponding function. This is an outstanding advantage of the Ritz method over all other practical methods of treating excited states, in which an indifferent approximate wave function can by chance lead to the correct value of the energy.

Among the results of computations on excited states of helium are the following: The lowest excited state of helium is a ^3S state. The observed energy is -4.3505 in the atomic units $R_H h$ while Hylleraas and Undheim,[10] using a six-term function, have computed an energy of -4.3504, an error of about one part in 40,000. The second excited state is a ^1S state, with an observed energy of -4.2920, while Coolidge and James[12] have used an eleven-term function to compute the value -4.2916, in error by one part in 10,000. The greater complexity of the function needed in the case of the more highly excited state will illustrate a general feature of computations with the Ritz method: convergence on the correct function becomes slower the more highly excited is the state considered. This factor, together with the increasing usefulness of certain approximate methods, limits the interest in extending Ritz computations to very highly excited states.

Another simple and important problem is that of the hydrogen molecule, H_2. This provides illustrations of the results obtainable with the Ritz method in rather different and more complicated circumstances, and also material for some general remarks about the method, which can here be based on more extensive studies than in the case of helium.

The problem of treating H_2, a system containing two heavy nuclei and two light electrons, can be reduced to that of solving the wave equation for the motion of the electrons in the field of the nuclei, considered as fixed. The wave equation is then

$$\left\{-\frac{h^2}{8\pi^2 m}(\nabla_1^2+\nabla_2^2)+\left(-\frac{e^2}{r_1}-\frac{e^2}{s_1}-\frac{e^2}{r_2}-\frac{e^2}{s_2}+\frac{e^2}{r_{12}}+\frac{e^2}{R}\right)\right\}\psi(R;\mathbf{r}_1,\mathbf{r}_2)$$
$$=E(R)\psi(R;\mathbf{r}_1,\mathbf{r}_2).$$

This partial differential equation involves the six coordinates of the two electrons as independent variables, and the fixed separation of the nuclei, R, as a parameter. It contains terms corresponding to the

[10] E. A. Hylleraas and B. Undheim, Zeitschrift für Physik, vol. 65 (1930), p. 759.
[11] J. K. L. MacDonald, Physical Review, vol. 43 (1933), p. 830.
[12] A. S. Coolidge and H. M. James, Physical Review, vol. 49 (1936), p. 676.

kinetic energies of the first and second electrons, the potential energy of the first electron at a distance r_1 from the first nucleus and s_1 from the second nucleus, the potential energy of the second electron in the field of the nuclei, the potential energy of repulsion of the two electrons, and the potential energy of repulsion of the two nuclei. Again the difficulty in solving the wave equation arises from the non-separability of the variables. In this case at most one variable can be separated out, and every exact solution depends on at least five variables —for instance, r_1, r_2, s_1, s_2, r_{12}.

The physical considerations on which one can base approximate solutions of this equation are interesting but so extensive and complicated that I cannot undertake to present them here. I can suggest only one of several possible lines of thought. If, for instance, one neglects the repulsion of the electrons the problem becomes completely separable. To this approximation, each electron moves as it would in the absence of the second electron—that is, as if it were the one electron of an ionized hydrogen molecule. Correspondingly the wave function becomes a product of two functions, each a solution of the wave equation for H_2^+.

$$\psi(R; r_1, r_2) = \chi_1(R; r_1)\chi_2(R; r_2),$$

$$\left\{-\frac{h^2}{8\pi^2 m}\nabla_1^2 - \frac{e^2}{r_1} - \frac{e^2}{s_1}\right\}\chi(R; r_1) = \epsilon(R)\chi(R; r_1).$$

Now this latter equation is separable in elliptic coordinates $\lambda_1=(r_1+s_1)/R$, $\mu_1=(r_1-s_1)/R$, ϕ_1=azimuth about the internuclear axis, and the form of the solutions can be determined in detail. In the cases to be mentioned later the interesting solutions are independent of ϕ and can be well approximated by a sum of a few terms of the form $e^{-\delta\lambda}\lambda^m\mu^j$. This suggests that one might well form the terms of a Ritz function for hydrogen as products of such factors, one for each electron, multiplied by various powers of the interelectronic distance to take account of the tendency of the wave function to decrease as the electrons approach each other: $e^{-\delta(\lambda_1+\lambda_2)}\lambda_1^m\lambda_2^n\mu_1^j\mu_2^k r_{12}^p$. The symmetry or antisymmetry required of the function according as one treats singlet or triplet states of the system can be assured with only slight complication of the form of the terms: $e^{-\delta(\lambda_1+\lambda_2)}(\lambda_1^m\lambda_2^n\mu_1^j\mu_2^k \pm \lambda_1^n\lambda_2^m\mu_1^k\mu_2^j)r_{12}^p$. Coolidge and I[13] have used just such terms in treating the ground state of H_2. For application to excited states of the molecule only one slight modification is necessary. Since the electrons are no longer

[13] H. M. James and A. S. Coolidge, Journal of Chemical Physics, vol. 1 (1933), p. 825; ibid., vol. 3 (1935), p. 129.

equivalent, one must introduce two different exponential parameters for the two electrons, corresponding to the different effective nuclear charges.

In working with these functions we have made extensive studies of the way in which the computed energy converges on the correct energy as more and more terms are added to the Ritz function. The convergence behavior of course varies with the chosen form of the terms, but when a physically appropriate choice has been made the convergence follows a remarkably similar pattern, whether one is working with some state of the hydrogen molecule, or the helium or lithium atoms, or the lithium molecule Li_2, or any other system with which we have had any experience.

In a favorable case, of which the ground state of H_2 is typical, the convergence is very simple and regular. A presentable approximate function and energy value can be obtained with relatively few terms —say five or six. Thereafter one finds that the small improvements obtained on adding further terms to the function are nearly additive. Thus a term which produces an improvement of .01 volt in the computed energy when it is added to a five-term function will give nearly the same improvement when added, separately or together with other terms, to a ten-term function. Deviations from strict additivity of improvements are almost invariably in the direction of a reduction in the importance of a term when it is added to a better function. Thus it is possible to put an upper limit on the error arising from neglect of any given term by trying it out together with a small number of others. This greatly reduces the labor of eliminating useless terms from consideration and getting the simplest possible Ritz function of a given accuracy.

Another important feature of the convergence is the rapid and regular falling off in importance of the terms in a sequence in which the value of any single exponent is increased. Thus in one case the term with exponents (0, 0, 1, 0, 0) was one of the most important terms, (0, 1, 1, 0, 0) gave an improvement of .008 electron volts, and (0, 2, 1, 0, 0) an improvement of .0015 electron volts. This rapid convergence makes it relatively easy to estimate closely the improvement in energy to be obtained on adding to the Ritz function all the terms of a given sequence. The near independence of the improvements due to the different sequences then makes possible a useful estimate of the convergence limit as all terms of the complete set are added to the Ritz function. For example, we have obtained the following results in treating the ground state of H_2. A five-term function gave the energy of the molecule as -2.3329 atomic units, against the observed value

of -2.3493—a result already far more accurate than has been obtained by any method other than the Ritz method. A thirteen-term function gave the energy as -2.3470, correct to 1/10 per cent; the root-mean-square error in that function itself is of the order of 1 per cent. Finally, a careful convergence estimate led to the energy value $-2.3492 \pm .0008$. When this value was published it was considerably more reliable than any available experimental result, but a later very accurate determination by Beutler and Jünger[14] gave the experimental value quoted above, in complete agreement with the apparent convergence limit of the Ritz method.

In treating some of the excited states of H_2 we have encountered a less simple convergence behavior.[15] Sometimes it happens that two terms will cooperate; that is, they will together improve the computed energy by more than the sum of the improvements they would produce separately. We have not encountered any case in which three or more terms have formed a cooperating group. When such cooperation occurs it is usually easy to understand in physical terms. In practice in treating a given problem one quickly finds out whether such cooperation is likely to occur and can predict fairly accurately what pairs of terms will show it. Somewhat more care is required in discarding terms and making convergence estimates, but the difficulty is not serious.

Somewhat more annoying is the general tendency toward slower convergence in the case of the excited states. With the ground state five terms suffice to give a very useful function, while with an excited state ten terms may be necessary to obtain similar accuracy. This increasing slowness of convergence is most marked when one deals with the higher states of a given symmetry. Thus our best function for the ground state, a ${}^1\Sigma_g$ state, contains thirteen terms. To get a somewhat less accurate result for the first excited state of this character sixteen terms were needed, and we have never succeeded in getting a usable approximation for the second excited ${}^1\Sigma_g$ state, which, incidentally, would be of particular physical interest.

Despite these difficulties the Ritz method has yielded very useful results for several excited states of H_2.[15] The maximum accuracy for practical work was obtained with functions containing about sixteen terms, after many other terms were tried and discarded as negligible. These functions are accurate to the order of 2 per cent, the computed energies to about 0.1 per cent.

[14] H. Beutler and H.-O. Jünger, Zeitschrift für Physik, vol. 101 (1936), p. 304.

[15] H. M. James, A. S. Coolidge and R. D. Present, Journal of Chemical Physics, vol. 4 (1936), p. 187; A. S. Coolidge and H. M. James, ibid., vol. 6 (1938), p. 730.

The physical significance of such applications of the Ritz method may be illustrated by the results of a computation in which use was made of these wave functions.[16] When H_2 is bombarded with electrons having more than 12 electron volts of energy it emits a strong continuous spectrum in the ultraviolet. This arises when molecules excited to the stable $1s\sigma 2s\sigma\ {}^3\Sigma_g$ state pass over into the lower and unstable $1s\sigma 2p\sigma\ {}^3\Sigma_u$ state. If the energy of the bombarding electrons is carefully kept to a minimum or if enough helium is present in the discharge this spectrum takes on a particularly simple and reproducible character, for the vibrations of the molecule in the initial electronic state are suppressed. Using Ritz wave functions for both electronic states of the molecule we have made a complete theoretical prediction of the intensities in this spectrum. The results appear in Figure 1, together with the observational results of Smith, the best

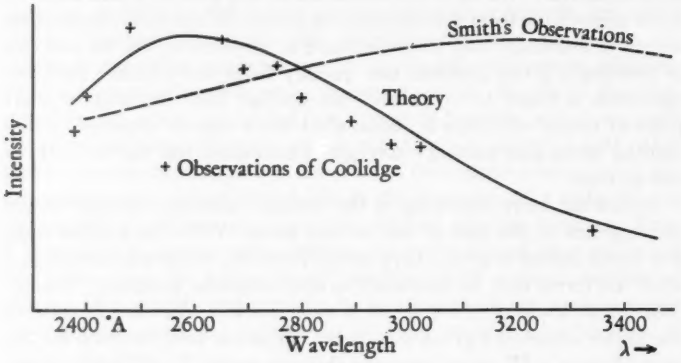

FIG. 1

available at the time. The agreement was not to be regarded as satisfactory. Since we had complete confidence in the theory we were forced to the conclusion that the experiments were in error, and that the intensity standards used by Smith were not satisfactory for work in this very difficult region of the spectrum. This has now been proved to be correct by the preliminary results of some very careful experimental work by Dr. Coolidge, also shown in Figure 1. On employing improved intensity standards, he finds that the gross disagreement of theory with experiment disappears. It is true that there remain discrepancies which are larger than one would like. This seems to be due

[16] H. M. James and A. S. Coolidge, Physical Review, vol. 55 (1939), p. 184.

to difficulties with the intensity standard, which is still less reliable than the theoretical computations. It would not be at all unreasonable to work backward, from the theory and the observations of Coolidge, to an improved calibration of the standard source.

Now a somewhat less satisfactory aspect of this subject must be mentioned. While the helium atom and the hydrogen molecule are of rather special interest to physicists they are also almost the simplest systems of their respective types, much too simple to be entirely typical. Unfortunately one soon finds on attempting to extend the Ritz method to the treatment of other atomic systems that the difficulties are numerous and serious. To indicate their character I need only to pass to the case of lithium.

The lithium atom consists of a nucleus and three electrons. In the simplest possible case the wave functions depend on six essential and non-separable variables—say the six distances between pairs of particles. The first discouraging effect of this greater number of variables is the corresponding increase in the number of potentially important Ritz terms with all exponents small. In passing from the hydrogen molecule with five essential variables to the lithium atom with six there is actually an increase from eight to thirty-two of the Ritz terms in which no exponent exceeds 1. The number of sequences of terms to be investigated in making convergence estimates is correspondingly increased.

Each term is also more difficult to work with. The increased number of electrons makes the symmetry conditions more complicated,[17] so that each Ritz term is, at the simplest, the sum of six parts, instead of the two parts needed with helium or hydrogen. There are thus more integrals to be evaluated in setting up the secular equation for the problem, and they are more difficult to evaluate because of the greater number and the awkward character of the variables of integration.

In the case of lithium it is possible to overcome these difficulties.[18] The observed total energy of the lithium atom is -14.9578 atomic units. Using a 17-term function we have computed the energy -14.9566, in error by one part in 10,000. Of more physical interest, perhaps, is the energy needed to ionize the atom. Here the experimental result is 5.364 e.v., while our convergence estimate leads to the theoretical value $5.363 \pm .007$ e.v.

It does not seem useful, however, to apply the Ritz method in accurate treatments of atoms more complicated than lithium. The diffi-

[17] H. M. James and A. S. Coolidge, Physical Review, vol. 55 (1939), p. 873.
[18] H. M. James and A. S. Coolidge, Physical Review, vol. 49 (1936), p. 688.

culties increase enormously and dampen effectively any interest one may have in this method of studying the systems. In the case of molecules the difficulties are even greater; to give an accurate treatment of the molecule Li_2 is much more than twice as difficult as to give a corresponding treatment of the lithium atom. It is true that the Ritz method has been fruitfully applied to the treatment of LiH,[19] and of Li_2,[20] and especially of the ionized molecules[19] LiH^+ and Li_2^+,[21] but here again the limits to the extension of the method seem to have been reached.

In conclusion I should emphasize that this particular field of atomic physics has been discussed, not as a summary of all that has been accomplished in physics using the Rayleigh-Ritz method, but rather as an illustration of how the method is likely to work out in practice where it is applied. Applications have been made in widely varied fields—the theory of nuclear structure, acoustics, the electromagnetic theory of cavity resonators—to mention three. The field I have discussed is typical in many respects. One can expect in general that the Ritz method can yield extremely accurate results with some problems not tractable to other methods of computation, and that its application will become impracticable just in cases where very many independent variables must be considered, or boundary or other conditions are especially complicated. What is abnormal about this field is the relative completeness with which the Ritz method has been exploited in it. This is due just to the scarcity of atomic problems in which one can deal with a reasonably small number of independent variables. In the majority of fields one can expect to find an unending sequence of problems to which the method can be usefully applied.

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[19] J. K. Knipp, Journal of Chemical Physics, vol. 4 (1936), p. 300.
[20] H. M. James, Journal of Chemical Physics, vol. 2 (1934), p. 794.
[21] H. M. James, Journal of Chemical Physics, vol. 3 (1935), p. 9.

SOME APPLICATIONS OF CERTAIN POLYNOMIAL CLASSES[1]

I. M. SHEFFER

The term *applications* will be construed broadly enough as to include *properties*. It is here proposed, then, to examine some properties and some applications of some classes of polynomial sets.

1. **Formal properties of Appell sets.** The power series approach to the theory of analytic functions presents us with a very simple polynomial set.[2] Thus, if $f(x)$ is analytic about $x=a$, we have the expansion

$$f(x) = \sum_{n=0}^{\infty} c_n \frac{(x-a)^n}{n!}, \qquad c_n = f^{(n)}(a), \tag{1}$$

in terms of the set

$$\{(x-a)^n/n!\}.$$

This set has the important property of reproducing itself under the operation of differentiation, in accordance with the rule.

$$\frac{d}{dx} P_n(x) = P_{n-1}(x), \tag{2}$$

where

$$P_n = (x-a)^n/n!.$$

Now this set is not uniquely determined by (2). There are in fact infinitely many sets of polynomials $\{P_n\}$ that satisfy (2). These bear the name of *Appell sets*,[3] after the man who in 1880 [1][4] first made a study of them. Appell sets will be the first class to be considered here.

There are many conditions that are equivalent to the defining relation (2) for Appell sets. Among the simplest are the following two:

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[1] An address delivered before the meeting of the Society in Washington, D.C., May 3, 1941, by invitation of the Program Committee.

[2] By a set $\{P_n\}$ we understand an infinite sequence $P_0, P_1, \cdots$, with P_n of degree n.

[3] Strictly speaking, Appell's definition is

$$\frac{d}{dx} P_n(x) = nP_{n-1}(x), \tag{2'}$$

but (2) is preferable for our purpose. If $\{P_n\}$ satisfies (2), then $\{n!\, P_n\}$ satisfies (2'), and conversely. A bibliography on Appell polynomials is given in Davis [5].

[4] Numbers in brackets refer to the bibliography placed at the end.

(i) *A set* $\{P_n\}$ *is an Appell set if and only if constants* $\{a_n\}$ *exist* $(a_0 \neq 0)$ *such that*

$$(3) \qquad P_n(x) = a_0 \frac{x^n}{n!} + a_1 \frac{x^{n-1}}{(n-1)!} + \cdots + a_n \frac{x^0}{0!}, \quad n = 0, 1, \cdots.$$

(This is the explicit representation of an Appell set.) (ii) *In order that* $\{P_n\}$ *be an Appell set it is necessary and sufficient that a formal power series*

$$(4) \qquad A(t) = \sum_0^\infty a_n t^n$$

exist so that (formally)

$$(5) \qquad A(t)e^{tz} = \sum_0^\infty P_n(x)t^n.$$

$A(t)$ will be called the *generating function* for $\{P_n\}$.

It is natural to inquire, regarding a set of polynomials, if it has properties analogous to those of the classical sets (Legendre, Hermite, and so on). For example, does it satisfy a linear differential equation of the form

$$L_0 y + L_1 y' + L_2 y'' = \lambda y$$

where L_i is a polynomial of degree not exceeding i, and λ is a parameter which takes on a value λ_n for the nth polynomial of the set? Or an equation of the same type but of higher order? Such equations contain too few constants to serve for *all* Appell sets, although we shall see that there is a subclass of such sets that satisfy equations of this type. If, however, we are willing to permit the order to become infinite:

$$(6) \qquad \sum_{r=0}^\infty L_r(x) y^{(r)}(x) = \lambda y(x),$$

where L_r is of degree not exceeding r, then there is ample freedom not only for Appell sets but for all sets. That is, (6) is *a universal equation* for all polynomial sets in that given an *arbitrary* set $\{P_n\}$, it is possible to choose $\{L_r\}$, $\{\lambda_r\}$, and indeed in infinitely many ways, so that $\{P_n\}$ satisfies (6) $(y = P_n$ for $\lambda = \lambda_n)$.

The advantages of a universal or, as we may say, *canonical* equation are evident. But this very generality may sometimes be a defect in studying particular sets, unless it is possible to characterize the coefficients $\{L_r\}$ corresponding to such sets. Fortunately a charac-

terization can be achieved in the case of Appell sets. The simplest (but not the only[5]) canonical equation of form (6) for Appell sets[6] is [11]

$$(7) \qquad (b_0 + x)P_n' + b_1P_n'' + b_2P_n''' + \cdots = nP_n, \quad n = 0, 1, \cdots.$$

That is, $L_0=0$, $L_1=b_0+x$, $L_k=b_{k-1}$, $k\geqq 2$; $\lambda_n=n$. Moreover, if $\{P_n\}$ has the generating function $A(t)$, then $\{b_n\}$ is defined by

$$(8) \qquad \sum_0^\infty b_nt^n \equiv B(t) = A'(t)/A(t).$$

It follows from (7) that $\{P_n\}$ *is an Appell set if and only if a sequence* $\{b_n\}$ *exists so that*

$$(9) \quad nP_n(x) = (b_0 + x)P_{n-1} + b_1P_{n-2} + \cdots + b_{n-1}P_0, \quad n = 1, 2, \cdots.$$

This is of interest because it gives the set $\{P_n\}$ by recurrence. It has been used by Webster [17] to give a simple proof that the Hermite polynomial set is the only Tchebycheff (orthogonal) set that is at the same time an Appell set. Here then is an Appell set that satisfies a second order equation of form (6). In general we have the theorem: *An Appell set* $\{P_n\}$, *with generating function* $A(t)$, *satisfies a finite order equation of type* (6) *if and only if* $A(t)$ *has the form*

$$(10) \qquad A(t) = e^{Q(t)}, \qquad Q(t) = \textit{polynomial};$$

and the minimum order of all such equations for $\{P_n\}$ *is the degree*[7] *of* $Q(t)$ [11]. As examples, the set $\{(x-a)^n/n!\}$ and the Hermite set have for $A(t)$ the respective functions e^{-at}, $e^{-t^2/4}$; and these sets satisfy equations of the first and second order respectively.

2. **Relation of Appell sets to functional equations.** In 1888 Pincherle made a study of the following linear difference equation with constant coefficients:

$$(11) \qquad L[y(x)] \equiv \sum_{j=1}^{k} a_jy(x + \omega_j) = F(x).$$

His memoir, in a French translation, was reprinted in 1926 in Acta Mathematica [10]. In the course of his investigation he pointed out the value of Appell sets. Thus for (11), two Appell sets preëminently suggest themselves, namely $\{P_n\}$, $\{Q_n\}$, defined respectively by

[5] There are infinitely many.

[6] Conversely, if $\{P_n\}$ satisfies (7), then $\{c_nP_n\}$ will be an Appell set for proper choice of the constants $\{c_n\}$.

[7] The one exception is when $Q(t)\equiv$constant.

$$L[x^n/n!] = P_n(x), \qquad L[Q_n(x)] = x^n/n!. \tag{12}$$

For as Pincherle observed, if $F(x)$ has the expansion

$$F(x) = \sum_0^\infty c_n P_n(x), \tag{13}$$

then a formal solution of (11) is given by

$$y(x) = \sum_0^\infty c_n x^n/n!; \tag{14}$$

and if $F(x)$ has the expansion (1) (for $a=0$), then

$$y(x) = \sum_0^\infty c_n Q_n(x) \tag{15}$$

formally satisfies (11). For the case that $F(x)$ is an entire function of finite exponential type,[8] Pincherle showed that there is a solution of (11) of the same kind. He also found a solution when $F(x)$ is analytic at infinity. His method of proof however did not involve polynomial sets.

In discussing the equation

$$L[y(x)] \equiv y(x+1) - y(x) = F(x), \tag{16}$$

a particular case of (11), A. Hurwitz [7] used the set $\{Q_n\}$, which for (16) is the set of Bernoulli polynomials. Since series in these polynomials serve to define only a limited class of entire functions, whereas he was interested in the case where $F(x)$ is an arbitrary entire function, Hurwitz modified the polynomials by adding to them linear combinations of properly chosen exponential functions. The device used has been termed the *method of expanding contours*. For from the generating relation

$$\frac{te^{tz}}{e^t - 1} = \sum_0^\infty Q_n(x)t^n \tag{17}$$

for the Bernoulli set, valid for $|t| < 2\pi$, follows the formula

$$Q_n(x) = \frac{1}{2\pi i}\int_C \frac{e^{tz}dt}{(e^t - 1)t^n}, \tag{18}$$

C being a contour around $t=0$ and lying in $|t| < 2\pi$. If now C is expanded so as to include a certain number of poles of the integrand, the

[8] A definition will be given later.

new contour, call it C_n, will define for each $n=0, 1, \cdots$ a function

$$Q_n^*(x) = \frac{1}{2\pi i}\int_{C_n} \frac{e^{tz}dt}{(e^t-1)t^n}, \tag{19}$$

which is $Q_n(x)$ augmented by certain exponentials (the residues at the poles lying within C_n). Hurwitz proved that to every entire function $F(x)$ given by (1) ($a=0$) there is an entire function solution of (16) given by (15) with Q_n replaced by Q_n^*.

Carmichael [4] extended the method and the conclusion to equation (11). The method of expanding contours can be shown to apply to (11) and (16) even when $F(x)$ is not an entire function. All that is required is that $F(x)$ be analytic about some point in a circle of radius exceeding a number r, whose (fixed) value is determined by the equation and the method. That is, one obtains what may be called a *semi-local* solution rather than a completely local one.

This restrictive condition on $F(x)$ can be, and is, lifted by emphasizing the set $\{P_n\}$ rather than $\{Q_n\}$ [12], [13]. For equation (11), $P_n(x)$ is given by

$$P_n(x) = \frac{1}{n!}\left[a_1(x+\omega_1)^n + \cdots + a_k(x+\omega_k)^n\right]. \tag{20}$$

Let the points $-\omega_j, j=1, \cdots, k$, be plotted. For each x let

$$\rho_j(x) = |x+\omega_j|, \qquad \rho(x) = \max\{\rho_1(x), \cdots, \rho_k(x)\}.$$

As x varies, there is a unique point x^* at which $\rho(x)$ attains its minimum value $\rho=\rho^*$. This means that the circle of radius ρ^* and center x^* will cover the set $\{-\omega_j\}$, and that no other circle of equal or smaller radius will have this property.

Now consider (20). It is undesirable, from the point of view of convergence, to consider those values of x for which the maximum $\rho(x)$ is attained by two or more of the $\rho_j(x)$'s. The set of such points x, which we shall term the *critical set*, is to be excepted. The following results can be proved: *If the series* $\sum n!c_nP_n(x)$ *converges at* $x=x_0$ (*not in the critical set*), *then it converges absolutely in the circular polygon*

$$\rho_j(x) = |x+\omega_j| < \rho(x_0), \qquad j=1, \cdots, k, \tag{21}$$

and uniformly in every closed region therein; *and a necessary and sufficient condition that the above series converge for at least one* x *not in the critical set is that* $\limsup |c_n|^{1/n} < 1/\rho^*$.

When such a series converges, and thus converges uniformly in some region, the function defined by the series is of course analytic.

The converse is also true: *If and only if a function $F(x)$ is analytic at $x=x^*$ can it be expressed as a convergent P_n-series.* The manner of proof is roundabout. Set $F(x)=1/(x-\alpha)$ in (11), α being a parameter. The resulting equation

$$L[W(x;\alpha)] = \frac{1}{x-\alpha} \tag{22}$$

is shown to have two (or three, according to the case) meromorphic solutions, from which follow two (or three) P_n-expansions for $1/(x-\alpha)$, valid in some neighborhood of $x=x^*$, for α on two (or three) arcs that form a closed contour around x^*. Application of the Cauchy integral formula then shows that $F(x)$, analytic at x^*, also has a convergent P_n-series. It follows, also, that *if $F(x)$ is analytic about x^* there is an analytic solution of* (11) *in the neighborhood of x^**; and a translation of the variable x permits the point x^* to be replaced by any other point of the plane.

It should be added that Appell expansions are usually *not* unique; that is, that the function zero possesses at least one P_n-expansion in which the coefficients are not all zero. To see this we turn back to (5). If $t_0 \neq 0$ is a zero of the generating function $A(t)$, then

$$0 = \sum_0^\infty t_0^n P(x) \tag{23}$$

is an expansion of zero.

We now consider the more general functional equation

$$A[y(x)] \equiv \sum_{k=0}^{\infty} a_k y^{(k)}(x) = F(x), \tag{24}$$

which is a linear differential equation of infinite order, with constant coefficients. Equation (11) is subsumed under (24). For if we call

$$A(t) = \sum_0^\infty a_k t^k \tag{25}$$

the *generating function for operator $A[y]$*, then (11) is that case of (24) for which

$$A(t) = \sum_{j=1}^{k} a_j e^{\omega_j t}. \tag{26}$$

The Appell set $\{P_n\}$ defined by (12) likewise has as its generating function the $A(t)$ of (25). Now the expansion problem will differ with the character of $A(t)$. In general, the larger the class of functions that

we wish to expand in a $\{P_n\}$-series, the greater must be the restriction on $A(t)$. A fruitful assumption, and one that we now make, is that $A(t)$ is of *finite exponential type.*[9] That is, the coefficients of (25) satisfy the inequality

$$\limsup_{n\to\infty} |n!a_n|^{1/n} < \infty. \tag{27}$$

If this superior limit has the value σ, then $A(t)$ is said to be of *exponential type* σ (*exp. type* σ).

From (6) we see that $A(t)e^{tz}$ is, in the variable t, of exp. type not exceeding $\sigma+|x|$, so that $\limsup |n!P_n(x)|^{1/n} \leqq \sigma+|x|$. This is, however, too crude an estimate. To better it, we observe that if a function $C(t)=\sum_0^\infty c_n t^n$ is of exp. type σ, then it is the *associated entire function* (*AEF*) relative to the function $C_1(t)=\sum_0^\infty n!c_n t^n$; and this latter function has $1/\sigma$ as its radius of convergence. Conversely, if by studying $C_1(t)$ one can locate the singularity nearest the origin, that will serve to determine the type of $C(t)$.

Applying this principle to (5), it is found [14] on setting

$$A(t) = \mathrm{AEF}\{A_1(t)\}, \qquad A^*(t) = tA_1(t), \tag{28}$$

that

$$A^*\left(\frac{1}{t-x}\right) = \sum_0^\infty \frac{n!P_n(x)}{t^{n+1}}, \tag{29}$$

$|t|$ sufficiently large. Let $G=\{\alpha\}$ be the set of all the singularities of $A^*(t)$ in the whole plane, and define the continuous function $D(x)$ by

$$D(x) = \max_G |x + 1/\alpha|. \tag{30}$$

It can then be established that

$$\limsup |n!P_n(x)|^{1/n} = D(x). \tag{31}$$

Let us call the curves $D(x)=c$ the *level curves* for the set $\{P_n\}$. A point set discussion reveals the following information: There is a unique point x^* where $D(x)$ has its minimum value D_m. For every $c>D_m$, the level curve is a *simple closed convex curve*, containing in its interior the level curves $D(x)=c'$ for every c' in $D_m \leqq c'<c$. In the case that the points $\{\alpha\}$ are finite in number, the level curves consist of circular arcs. This is the situation relative to equation (11), where

[9] See Muggli [9], where other cases are also taken up. The solution of (24) is there carried out by the method of expanding contours, and for the case where $A(t)$ is of finite exponential type, $F(x)$ is assumed to be an entire function.

the region of convergence of a P_n-expansion was found to be a circular polygon.

With respect to the present general case we find that: *If* $\limsup |c_n|^{1/n} = \rho < 1/D_m$, *then series* $\sum n! c_n P_n(x)$ *converges absolutely interior to the level curve* $D(x) = 1/\rho$, *and converges uniformly in every closed region therein, thus representing an analytic function there.*

At this point we are forced to abandon the method that proved so useful in dealing with (11), for we have not sufficient knowledge of solutions of the equation corresponding to (22) to follow the earlier plan. Turning to the polynomial set $\{Q_n\}$ of (12), and modifying it by the method of expanding contours, it is possible, first to solve equation (24) *semi-locally*, and then to obtain a P_n-expansion theorem: *If* $F(x)$ *is analytic about a point* $x = x_0$ *in a circle of radius exceeding* λ, *where* λ *is determined by the operator* A, *then equation* (24) *has an analytic solution* $y(x)$ *in some neighborhood of* x_0; *and if* $F(x)$ *is analytic about* $x = 0$ *in a circle of radius exceeding* λ^* (*again determined by operator* A), *then* $F(x)$ *has a convergent Appell expansion*

$$F(x) = \sum_0^\infty n! c_n P_n(x). \tag{32}$$

The numbers λ, λ^* are not sharply defined. Burdette [3] has obtained sharp values on making the additional hypothesis that $A(t)$ is bounded on a certain infinite sequence of expanding contours.

3. **Sets of type zero.** Let us turn back to equation (16). Relative to it one can consider the set of Newton polynomials

$$N_n(x) = \frac{x(x-1)\cdots(x-n+1)}{n!}; \tag{33}$$

for this set, which is not an Appell set, has the fundamental property of reproducing itself under application of the difference operator:

$$\Delta N_n(x) = N_{n-1}(x), \tag{34}$$

so that if $F(x)$ has the expansion

$$F(x) = \sum_0^\infty c_n N_n(x), \tag{35}$$

then a formal solution of (16) is given by

$$y(x) = \sum_0^\infty c_n N_{n+1}(x). \tag{36}$$

We may say that $\{N_n\}$ is a set of *difference polynomials*, as is every other set (and there are infinitely many such) that has the property (34). Such sets may be examined as were Appell sets, but at this point there is suggested a general problem: Can there be assigned to an arbitrary set $\{P_n\}$ an operator J that will reproduce $\{P_n\}$ in the sense that

$$J[P_n(x)] = P_{n-1}(x) \tag{37}$$

for all n?

The answer is in the affirmative, and J can be expressed in the form

$$J[y] \equiv \sum_{n=1}^{\infty} L_{n-1}(x)y^{(n)}(x) \tag{38}$$

where $L_k(x)$ is a polynomial of degree not exceeding k. This too is a universal (or canonical) operator, since to *every* set corresponds a determined J. For Appell sets and for difference sets, the L_n's are constants. This suggests examination of all sets with constant L_n's. We shall say that *a set* $\{P_n\}$ *is of zero type* [15] if in its associated operator J (as given by (38)) the polynomials $L_n(x)$ are all constants.

For zero type sets, then, (38) can be written

$$J[y(x)] \equiv \sum_{n=1}^{\infty} c_n y^{(n)}(x), \qquad c_1 \neq 0. \tag{39}$$

Let the formal power series

$$J(t) \equiv \sum_{1}^{\infty} c_n t^n \tag{40}$$

be called the *generating function* for (39), and let $H(t)$ be the inverse power series to $J(t)$, so that (formally)

$$J(H(t)) = H(J(t)) = t. \tag{41}$$

Zero type sets can then be characterized as follows: $\{P_n\}$ *is a set of type zero if and only if a formal power series* $A(t) = \sum_0^\infty a_n t^n$ *exists so that*

$$A(t)e^{xH(t)} = \sum_{0}^{\infty} P_n(x)t^n. \tag{42}$$

Observe that the Laguerre polynomials $\{L_n(x)\}$ are of zero type, since

$$\frac{1}{1-t} \exp\left(\frac{-xt}{1-t}\right) = \sum_{0}^{\infty} L_n(x)t^n. \tag{43}$$

Various other characterizations of zero type sets can be found. One of the simplest is that *constants* $\{q_{k0}\}$, $\{q_{k1}\}$ *exist so that the recurrence relation*

$$\sum_{k=1}^{\infty} (q_{k0} + xq_{k1})P_{n-k}(x) = nP_n(x) \tag{44}$$

holds for $n = 1, 2, \cdots$. The relation between (42) and (44) is expressed by the series

$$\frac{A'(t)}{A(t)} = \sum_{0}^{\infty} q_{n+1,0}t^n, \qquad H'(t) = \sum_{0}^{\infty} q_{n+1,1}t^n. \tag{45}$$

Having learned that among Appell sets is a Tchebycheff set (the Hermite polynomials) one naturally asks if there are other orthogonal sets in the extended class of zero type sets. This problem was solved by Meixner [8], by use of the Laplace transformation. He took (42) as his definition of the polynomials that he was investigating. Following Meixner, Geronimus obtained some interesting properties of zero type sets. A second resolution of the Meixner problem can be made by the combined used of (44) and the characterizing recurrence relation

$$Q_n(x) = (x + \lambda_n)Q_{n-1}(x) + \mu_n Q_{n-2}(x), \qquad n = 1, 2, \cdots, \tag{46}$$

for orthogonal polynomials. For (46) to represent a set of type zero it is necessary and sufficient that

$$\lambda_n = a + bn, \qquad \mu_n = (n - 1)(c + dn), \tag{47}$$

with $c+dn \neq 0$ for $n > 1$. From this will follow the explicit determination of the required sets, as (otherwise) obtained by Meixner.

Consider an operator J of form (39). The equation (24) can be equally well expressed in terms of J:

$$A[y(x)] \equiv \sum_{k=0}^{\infty} a_k^* J^k[y(x)] = F(x), \tag{48}$$

where $J^k[y] \equiv J[J^{k-1}[y]]$, $J^0[y] \equiv y$. As such, we see that polynomial sets of zero type suggest themselves as a means of studying equation (24). This study, however, awaits more knowledge of the expansion properties of such sets.

4. **On equations with non-constant coefficients.** The functional equations already considered have been of the constant coefficient type, and it may be asked if zero type sets can be applied to more general equations. Such application has been made to the linear differ-

ential equation of finite order, and this points the way to other equations (where, however, the results are as yet only formal).

Consider the differential equation

$$L[y(x)] \equiv y^{(k)} + \rho_1(x)y^{(k-1)} + \cdots + \rho_k(x)y = F(x). \tag{49}$$

We seek [16] an operator $M[y]$ that is inverse to $L[y]$ in accordance with the relations

$$ML[y] \equiv y; \qquad LM[y] \equiv y. \tag{50}$$

The assumed form of $M[y]$ is

$$M[y] \equiv \sum_{n=-\infty}^{+\infty} M_n(x)y^{(n)}(x), \tag{51}$$

where

$$y^{(-n)}(x) = \int_{x_0}^{x} \cdots \int_{x_0}^{x} y(t)dt, \quad n = 1, 2, \cdots. \tag{52}$$
$$\text{(n-fold)}$$

It is found that (50) can be satisfied by choosing $M_n(x) = 0$, $n > -k$; $M_{-k}(x) = 1$; and the other M's by recurrence:

$$\begin{aligned} M_{-m}(x) &= C_{m,m-1}(x)M_{-(m-1)}(x) + C_{m,m-2}(x)M_{-(m-2)}(x) + \cdots \\ &\quad + C_{m,k}(x)M_{-k}(x), \qquad m > k. \end{aligned} \tag{53}$$

(The functions $C_{ij}(x)$ are determined by the coefficients $\rho_n(x)$ and their derivatives.) On setting

$$H(x, t) = M_{-k}(x)\frac{(x-t)^{k-1}}{(k-1)!} + M_{-(k+1)}(x)\frac{(x-t)^k}{k!} + \cdots, \tag{54}$$

a solution of (49) is given by

$$y(x) = \int_{x_0}^{x} H(x, t)F(t)dt. \tag{55}$$

$H(x, t)$ itself satisfies (in the variable x) the homogeneous equation

$$L[u] = 0, \tag{56}$$

as do also $\partial H/\partial t$, $\partial^2 H/\partial t^2$, $\cdots$, $\partial^{k-1}H/\partial t^{k-1}$; and these k functions form an independent set of solutions.

The series (54) has an interesting domain of convergence, but we must forego a discussion of this [16]. What is of importance for our present aim is the observation that the Appell polynomial set $\{(x-t)^n/n!\}$ is present (at least, beginning with $n = k-1$) in this

series. Any other Appell set would (formally at least) lead to the same result, in the following sense: If $\{P_n(x)\}$ is an Appell, set then the formal series

$$(57)\quad K(x,t) = M_{-k}(x)P_{k-1}(x-t) + M_{-(k+1)}(x)P_k(x-t) + \cdots$$

satisfies (56). The convergence problem for (57) has not been examined, although relation (31) is certainly applicable.

With (49) as a guide, we turn to the corresponding difference equation

$$(58)\quad L[u] \equiv \Delta^k u(x) + q_1(x)\Delta^{k-1}u(x) + \cdots + q_k(x)u(x) = F(x).$$

Here we look for an inverse operator $M(LM[u] \equiv ML[u] \equiv u)$ in the form

$$(59)\qquad M[u] \equiv \sum_{n=-\infty}^{+\infty} M_n(x)\Delta^n u(x),$$

where for negative n the "sum" is meant; that is, $\Delta^n[\Delta^{-n}[u]] = u$. We can again choose $M_n(x) = 0$ for $n > -k$; $M_{-k}(x) = 1$; after which $M_{-(k+1)}(x)$, $M_{-(k+2)}(x)$, $\cdots$ are determined by recurrence, in analogy with (53). And like (54) (or (57)) we set up the (formal) series

$$(60)\quad K(x,t) = M_{-k}(x)P_{k-1}(x-t) + M_{-(k+1)}(x)P_k(x-t) + \cdots$$

where $\{P_n\}$ is any *difference* set ($\Delta P_n = P_{n-1}$).

Convergence has not yet been examined. Formally, however, (60) satisfies in the variable x the homogeneous equation $L[u] = 0$, as do the functions ΔK, $\Delta^2 K$, $\cdots$, where the differences are taken with respect to the variable t.

5. **Remarks on null-functions.** We shall close with some remarks on the problem of null-functions (as we may call them). Let an infinite sequence of functionals $\{\mathfrak{M}_n\}$ be given. We assume them to be of the form

$$(61)\quad \mathfrak{M}_n[f(x)] = c_{n0}f(0) + c_{n1}f'(0) + c_{n2}f''(0) + \cdots, \quad n = 0, 1, \cdots,$$

supposed applicable to a given class of analytic functions. By *a null-function* we shall understand a function $f(x)$, not identically zero, for which

$$(62)\qquad \mathfrak{M}_n[f(x)] = 0, \qquad n = 0, 1, \cdots.$$

The determination of null-functions is linked to the problem of finding the *radius of completeness* of the sequence of functions $\{M_n(t)\}$, where

$$M_n(t) = c_{n0} + c_{n1}t + c_{n2}t^2 + \cdots ; \tag{63}$$

that is, the largest number r such that every function $f(t)$, analytic in $|t| < r$, has an expansion

$$f(t) = \sum_0^\infty f_n M_n(t) \tag{64}$$

convergent in $|t| < r$ and uniformly convergent in every closed region therein.[10]

An interesting subclass of (61) is that in which $\mathfrak{M}_n$ has the form

$$\mathfrak{M}_n[f] = c_{nn}f^{(n)}(0) + c_{n,n+1}f^{(n+1)}(0) + \cdots , \qquad c_{nn} \neq 0. \tag{65}$$

An example of this, namely when

$$\mathfrak{M}_n[f] = f^{(n)}(a_n), \qquad |a_n| \leqq 1, \qquad n = 0, 1, \cdots , \tag{66}$$

leads to the Takenaka problem [18] of determining the largest number λ for which it is true that no entire function of exp. type less than λ can be a null-function. It is known that $\lambda \geqq \log 2$, and it is believed that $\lambda = \pi/4$. The example

$$f(x) = \cos\frac{\pi x}{4} - \sin\frac{\pi x}{4}, \qquad a_n = (-1)^n,$$

shows that $\pi/4$ cannot be exceeded.

Corresponding to the above sequence $a_n = (-1)^n$ (and it may very well be true for every sequence for which $\{a_n\}$ is bounded), the following is true: Given any $\sigma > 0$, there is only a *finite number* of linearly independent null-functions of exp. type not exceeding σ. (As $\sigma \to \infty$, however, the number of independent null-functions is unbounded.) If, however, the sequence $\{a_n\}$ is not required to be bounded, and in other cases of the functionals $\{\mathfrak{M}_n\}$, there may be a continuous array of null-functions. This fact we shall illustrate by borrowing from the theory of sets of type zero. Consider, in fact, the set $\{P_n\}$ for which

$$e^{zH(t)} = \sum_0^\infty P_n(x)t^n; \tag{67}$$

or, letting $J(t)$ be the inverse of $H(t)$,

$$e^{xt} = \sum_0^\infty P_n(x)[J(t)]^n. \tag{68}$$

[10] For work in this direction, consult Whittaker [18], Gontcharoff [6] and Boas [2]. These works include further references.

Take $[J(t)]^n$ as the function $M_n(t)$ of (63), which thus determines the functionals $\mathfrak{M}_n[f]$.

A simple example is given by $H(t)=t=J(t)$, for then $\mathfrak{M}_n[f]=f^{(n)}(0)$, so that there is no null-function that is analytic at the origin. Now take $H(t)=\log(t+1)$, $J(t)=e^t-1$. One verifies that the function $f(x)=e^{\alpha x}(1-e^{2k\pi ix})$ is a null-function for every integer k and for *every value of* α. That is, there is a continuous spectrum of null-functions. This property is shared by functionals $\{\mathfrak{M}_n\}$ corresponding to other choices of $H(t)$.

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ON TRIGONOMETRICAL SERIES WHOSE COEFFICIENTS DO NOT TEND TO ZERO[1]

R. SALEM

Let $\sum \rho_n \cos(nx-\alpha_n)$ be a series (S) such that

$$\rho_n \geqq 0, \qquad \limsup \rho_n > 0. \tag{1}$$

It is well known that the set E of convergence of S is of measure zero, a result due to Cantor and Lebesgue. More recently Rajchman has proved that E is a sum of an enumerable sequence of H-sets, and so in particular (an H-set being closed and of measure zero) that E is of the first category.[2]

We propose to establish a more precise property of the sets E and to show the connection between these sets and sets of absolute convergence (that is, sets in which a trigonometrical series can converge absolutely without being absolutely convergent everywhere).

We propose to call, in memory of Rajchman, "set of the type R" any set E such that a series (S) exists which satisfies the condition (1) and converges in E. E being a sum of closed sets it is natural to investigate the properties of perfect sets of the type R. Let P be such a set. For every x belonging to P we must have $\lim \rho_n \cos(nx-\alpha_n)=0$. But by (1) there exists an infinite sequence of integers $\{n_k\}$ such that $\rho_{n_k}>r>0$. Hence,

$$\lim \cos(n_k x - \alpha_{n_k}) = 0. \tag{2}$$

We can assume, without loss of generality that the point $x=0$ belongs to P. Hence

$$\lim \cos \alpha_{n_k} = 0. \tag{3}$$

(2) and (3) give immediately

$$\lim \left| \sin n_k x \sin \alpha_{n_k} \right| = 0, \tag{4}$$

and as by (3)

$$\lim \left| \sin n_k x \cos \alpha_{n_k} \right| = 0 \tag{5}$$

we get, by adding (4) and (5),

$$\lim \left| \sin n_k x \right| = 0. \tag{6}$$

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[1] Presented to the Society, May 2, 1941.

[2] See Zygmund, *Trigonometrical Series*, Warsaw, 1935, pp. 267–270.

This implies, for every x belonging to P, $\lim \sin^2 n_k x = 0$. Let $F(x)$ be any non-decreasing bounded function, constant in each interval contiguous to P, but not everywhere. The sequence $\{\sin^2 n_k x\}$ being uniformly bounded, we have

$$\lim \int_0^{2\pi} \sin^2 n_k x \, dF = 0$$

or

$$\lim \int_0^{2\pi} \cos 2n_k x \, dF = F(2\pi) - F(0), \tag{7}$$

that is to say, not only the Fourier-Stieltjes cosine coefficients of rank $2n_k$ of dF do not tend to zero (even if F is continuous), but they tend to the greatest possible limit as $k \to \infty$. It is interesting to observe that the sequence $\{n_k\}$ is independent of the function F.

It has been proved[3] that if a perfect set P is a set of absolute convergence, then for every function F of the above-described type

$$\limsup \int_0^{2\pi} \cos 2nx \, dF = F(2\pi) - F(0).$$

Thus it appears that sets of the type R and sets of absolute convergence are closely connected.

The necessary condition (7) shows that the condition that a set of the type R should be a sum of H-sets is too lax. In particular, even a single H-set is not, in general, of the type R, as can be seen by the example of Cantor's ternary set C. This is an easy consequence of the necessary condition (6) and of the well known fact that the set of all possible sums $x+y$ where x and y belong to C fills the whole interval on which C is constructed.[4]

Moreover, the sum of two perfect sets of the type R may not be of the type R. This result which has been proved by Marcinkiewicz[5] for perfect sets of absolute convergence holds good for sets of the type R and is a consequence of Marcinkiewicz's argument and of the necessary condition (6).

Let now P be a perfect set such that the condition (7) is satisfied. We can find a function F, bounded, constant in every interval con-

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[3] See Salem, *The absolute convergence of trigonometrical series*, Duke Mathematical Journal, vol. 8 (1941), pp. 317–334.

[4] See Zygmund, loc. cit., p. 133.

[5] Travaux de la Société des Sciences et des Lettres de Wilno, Classe des Sciences Mathématiques et Naturelles, vol. 12 (1937).

tiguous to P, but increasing from one interval to another, and a sequence $\{m_k\}$ such that $\int_0^{2\pi} \sin^2 m_k x \, dF < 1/k^4$. Hence

$$\sum_1^\infty k^2 \int_0^{2\pi} \sin^2 m_k x \, dF < \infty.$$

Hence "almost everywhere" in P, that is to say, in a subset P_1 of P such that the variation of F over $P - P_1$ is zero, $\sum_1^\infty k^2 \sin^2 m_k x < \infty$. But

$$\sum_1^k |\sin m_k x| < \left(\sum_1^k \frac{1}{k^2}\right)^{1/2} \left(\sum_1^k k^2 \sin^2 m_k x\right)^{1/2};$$

hence $\sum |\sin m_k x|$ converges in P_1, and P is "almost everywhere" of the type R (and also, almost everywhere, a set of absolute convergence).

MONTREAL, CANADA

UNITARY SPACES WITH CORRESPONDING GEODESICS[1]

N. COBURN

1. **Introduction.** This paper is divided into three parts. In the first section, the notation and fundamental concepts of hermitian geometry are reviewed. The second section develops the equations of geodesic curves X_1 which depend on a *real parameter* (t) and which are imbedded in a unitary space of n-dimensions K_n. Our principal result is: The equations of such geodesics differ from the equations of geodesics in Riemannian space in that the former contain the torsion affinor. In the third section, we classify the connections of two unitary spaces K_n, $'K_n$ whose geodesics correspond. First, we find the necessary and sufficient conditions that two unitary spaces K_n, $'K_n$ both with symmetric connection shall have their geodesics in correspondence. This last problem is solved in exactly the same manner as the similar problem in Riemannian space.[2] Secondly, we prove that if K_n has torsion and $'K_n$ has no torsion (symmetric connection), then their geodesics do not correspond. The problem of determining all connections of unitary spaces K_n, $'K_n$ both with torsion whose geodesics correspond is left open.

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[1] Presented to the Society, January 1, 1941.

[2] L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, 1926, p. 131.

2. **Notation.**[3] Consider a real space of $2n$-dimensions X_{2n} whose coordinates are given by the real variables

$$x^\lambda, \ y^\lambda, \qquad \lambda, \mu = 1 \cdots n. \tag{2.1}$$

Into this X_{2n}, we introduce the complex coordinates given by

$$\begin{aligned} \xi^\lambda &= x^\lambda + iy^\lambda, \\ \xi^{\lambda^*} &= x^\lambda - iy^\lambda, \qquad i = (-1)^{1/2}. \end{aligned} \tag{2.2}$$

Since the Jacobian of this transformation $(-2i)$ does not vanish over X_{2n}, then ξ^λ, ξ^{λ^*} constitute a set of $2n$ independent variables which map the X_{2n}. In view of the fact that ξ^{λ^*} are complex conjugates to ξ^λ, we determine the points of X_{2n} by merely assigning complex numbers to ξ^λ. We say that the ξ^λ determine "points" which build a *complex* space of n-dimensions X_n (the above real topological X_{2n}). Let us denote partial derivatives by

$$\partial_\mu = \partial/\partial\xi^\mu, \qquad \partial_{\mu^*} = \partial/\partial\xi^{\mu^*}, \tag{2.3}$$

and let $\phi(\xi^\lambda, \xi^{\lambda^*})$ be an analytic function of the variables ξ^λ, ξ^{λ^*}. Then, by the composite function theorem, we obtain

$$\partial\phi/\partial x^\mu = \partial_\mu\phi + \partial_{\mu^*}\phi, \tag{2.4}$$

$$\partial\phi/\partial y^\mu = i\partial_\mu\phi - i\partial_{\mu^*}\phi. \tag{2.5}$$

Solving for $\partial_\mu\phi$, $\partial_{\mu^*}\phi$, we find

$$\partial_\mu\phi = 1/2(\partial\phi/\partial x^\mu - i\partial\phi/\partial y^\mu), \tag{2.6}$$

$$\partial_{\mu^*}\phi = 1/2(\partial\phi/\partial x^\mu + i\partial\phi/\partial y^\mu). \tag{2.7}$$

If the function $\psi(\xi^\lambda)$,

$$\psi(\xi^\mu) = u(x^\lambda, y^\lambda) + iv(x^\lambda, y^\lambda), \tag{2.8}$$

is analytic in the sense of Cauchy-Riemann, then

$$\begin{aligned} \partial u/\partial x^\lambda - \partial v/\partial y^\lambda &= 0, \\ \partial v/\partial x^\lambda + \partial u/\partial y^\lambda &= 0. \end{aligned} \tag{2.9}$$

By expanding the right-hand side of (2.7), we find that (2.9) is equivalent to

$$\partial_{\lambda^*}\psi = 0. \tag{2.10}$$

From our point of view, this equation follows directly from the fact

[3] J. A. Schouten and D. J. Struik, *Einfuhrung in die Neueren Methoden der Differentialgeometrie*, P. Noordhoff, Groningen, Batavia, vol. 2, 1938, p. 225.

that ξ^λ, ξ^{λ^*} are independent variables. Hence equation (2.9) merely serves to interpret the equation (2.10) from the point of view of complex variable. If ψ^* is the complex conjugate function to ψ,

$$\psi^* = u - iv, \tag{2.11}$$

then it is easily shown that (2.9) is equivalent to the so-called conjugate equation

$$\partial_\lambda \psi^* = 0. \tag{2.12}$$

We seek to generalize the idea involved in (2.12). Corresponding to any function $\phi(\xi^\lambda, \xi^{\lambda^*})$, let $\phi^*(\xi^{\lambda^*}, \xi^\lambda)$ denote the function obtained by replacing i by $-i$. From (2.2), we see that the variables ξ^λ are then replaced by ξ^{λ^*} and conversely. We call this function ϕ^*, the conjugate of ϕ. In the future, we shall indicate the validity of the conjugate by the abbreviation "conj." It is to be noted on the formal side, that in passing to the conjugate, all indices will be starred. The star of a starred quantity removes the original star.

Consider the allowable analytic coordinate transformations with nonvanishing Jacobian

$$\xi^\lambda = \xi^\lambda(\xi^{\lambda'}), \qquad \xi^{\lambda'} = x^{\lambda'} + iy^{\lambda'}. \tag{2.13}$$

The corresponding conjugate equations are

$$\xi^{\lambda^*} = \xi^{\lambda^*}(\xi^{\lambda^{*\prime}}), \qquad \xi^{\lambda^{*\prime}} = x^{\lambda'} - iy^{\lambda'}. \tag{2.14}$$

We now introduce the unit affinor whose intermediary components are

$$A_{\mu'}^{\lambda} = \partial\xi^\lambda/\partial\xi^{\mu'}, \qquad A_{\mu^{*\prime}}^{\lambda^*} = \partial\xi^{\lambda^*}/\partial\xi^{\mu^{*\prime}} \tag{2.15}$$

A vector $v^\lambda(\xi^\lambda, \xi^{\lambda^*})$ is said to be of the *first type* if

$$v^{\lambda'} = A_\lambda^{\lambda'\,\lambda} v^\lambda. \tag{2.16}$$

Associated with each such vector is its conjugate, or vector of the *second type* v^{λ^*}, with transformation law

$$v^{\lambda^{*\prime}} = A_{\lambda^*}^{\lambda^{*\prime}\,\lambda^*} v^{\lambda^*}. \tag{2.17}$$

The theory can be extended to affinors of any mixed valence.

Let us introduce a connection in X_n, by means of the n^3 quantities $\Gamma_{\mu\alpha}^\lambda$ which are functions of position. Then, we define the covariant derivative of a contravariant vector by

$$\delta v^\lambda = dv^\lambda + \Gamma_{\mu\alpha}^\lambda v^\alpha d\xi^\mu, \text{ conj.} \tag{2.18}$$

The $\Gamma^{\lambda}_{\mu\alpha}$ transform in a well known manner[4] under (2.13). We now write for a covariant vector

$$\delta w_\lambda = dw_\lambda - \Gamma^{\alpha}_{\mu\lambda} w_\alpha d\xi^{\mu}, \text{ conj.} \tag{2.19}$$

By expanding the ordinary differential of a vector, we obtain

$$dv^\lambda = d\xi^\mu \partial_\mu v^\lambda + d\xi^{\mu^*} \partial_{\mu^*} v^\lambda, \text{ conj.} \tag{2.20}$$

If we define the covariant derivative of v^λ, w^λ by means of

$$\nabla_\mu v^{\lambda} = \partial_\mu v^{\lambda} + \Gamma^{\lambda}_{\mu\alpha} v^{\alpha}, \qquad \nabla_\mu w_\lambda = \partial_\mu w_\lambda - \Gamma^{\alpha}_{\mu\lambda} w_\alpha, \tag{2.21}$$

$$\nabla_{\mu^*} v^{\lambda} = \partial_{\mu^*} v^{\lambda}, \qquad \nabla_{\mu^*} w_\lambda = \partial_{\mu^*} w_\lambda, \text{ conj.}, \tag{2.22}$$

then

$$\delta v^\lambda = d\xi^\mu \nabla_\mu v^\lambda + d\xi^{\mu^*} \nabla_{\mu^*} v^\lambda, \text{ conj.}, \tag{2.23}$$

$$\delta w_\lambda = d\xi^\mu \nabla_\mu w_\lambda + d\xi^{\mu^*} \nabla_{\mu^*} w_\lambda, \text{ conj.}, \tag{2.24}$$

An hermitian X_n, with covariant derivative defined by (2.21), (2.22) is denoted by K_n.

We now introduce an hermitian tensor with hermitian symmetry

$$a_{\lambda\mu^*} = (a_{\lambda\mu^*})^{*\prime} = a_{\mu^*\lambda}, \tag{2.25}$$

the ($'$) indicating the transpose matrix. If we condition $a_{\lambda\mu^*}$ by requiring that

$$\delta a_{\lambda\mu^*} = 0 = (\partial_\nu a_{\lambda\mu^*} - \Gamma^{\rho}_{\nu\lambda} a_{\rho\mu^*}) d\xi^{\nu} + (\partial_{\nu^*} a_{\lambda\mu^*} - \Gamma^{\rho^*}_{\nu^*\mu^*} a_{\lambda\rho^*}) d\xi^{\nu^*}, \tag{2.26}$$

then the space K_n is said to be a unitary K_n. For such a space, from (2.26), we can prove[5]

$$\nabla_\nu a_{\lambda\mu^*} = \partial_\nu a_{\lambda\mu^*} - \Gamma^{\rho}_{\nu\lambda} a_{\rho\mu^*} = 0, \text{ conj.}, \tag{2.27}$$

$$\nabla_{\nu^*} a_{\lambda\mu^*} = \partial_{\nu^*} a_{\lambda\mu^*} - \Gamma^{\rho^*}_{\nu^*\mu^*} a_{\lambda\rho^*} = 0, \text{ conj.} \tag{2.28}$$

The $a_{\lambda\mu^*}$ is now a fundamental tensor and can be used to raise and lower indices through the ∇ operator. If we define the contravariant fundamental tensor $a^{\sigma^*\kappa}$ by

$$a^{\sigma^*\kappa} a_{\lambda\sigma^*} = A^{\kappa}_{\lambda}, \text{ conj.}, \tag{2.29}$$

then (2.27), (2.28) may be solved for the connection

$$\Gamma^{\kappa}_{\mu\lambda} = (\partial_\mu a_{\lambda\mu^*}) a^{\sigma^*\kappa} = (\partial_\mu a_{\sigma^*\lambda}) a^{\kappa\sigma^*}, \tag{2.30}$$

$$\Gamma^{\kappa^*}_{\mu^*\lambda^*} = (\partial_{\mu^*} a_{\sigma\lambda^*}) a^{\kappa^*\sigma} = (\partial_{\mu^*} a_{\lambda^*\sigma}) a^{\sigma\kappa^*}. \tag{2.31}$$

[4] J. A. Schouten and D. J. Struik, loc. cit., p. 227.

[5] J. A. Schouten and D. J. Struik, loc. cit., p. 234.

Finally, we introduce the torsion affinor

$$S_{\mu\lambda}^{\cdot\cdot\kappa} = 1/2(\Gamma_{\mu\lambda}^{\kappa} - \Gamma_{\lambda\mu}^{\kappa}) = \Gamma_{[\mu\lambda]}^{\kappa}, \text{ conj.} \tag{2.32}$$

The sign [ ] means that the antisymmetric product of the enclosed indices is to be formed; the sign ( ) means that the symmetric product of the enclosed indices is to be formed.

3. **Equations of geodesics in K_n.** Consider the curve X_1 defined in K_n by means of the equations

$$\xi^{\lambda} = \xi^{\lambda}(t), \qquad \xi^{\lambda^*} = \xi^{\lambda^*}(t), \tag{3.1}$$

where (t) is a real parameter and ξ^{λ}, ξ^{λ^*} are conjugate functions. Evidently, such equations determine a curve X_1 in the real topological X_{2n}. We define the element of arc length along this X_1 by

$$ds = (a_{\lambda\mu^*} d_t\xi^{\lambda} d_t\xi^{\mu^*})^{1/2} dt. \tag{3.2}$$

By substituting (3.1) into (3.2) and integrating, we obtain (s) the arc length parameter. It is to be noted that this parameter exists only because of the fact that (t) is real. By use of (2.25) and the conjugate of (3.2), we see that (ds) and hence (s) are real.

We follow the well known methods of the calculus of variations[6] in finding the geodesics of K_n. Our problem is to find that X_1 for which the first variation of

$$I = \int_{t_0}^{t_1} (a_{\lambda\mu^*} d_t\xi^{\lambda} d_t\xi^{\mu^*})^{1/2} dt \tag{3.3}$$

is zero. Let ξ^{λ} represent the coordinates of a point on the geodesic X_1; let $'\xi^{\lambda}$ represent the coordinates of a point on the varied $'X_1$; let ϵ be an infinitesimal; and let ω^{λ} be an arbitrary function of ξ^{λ}, ξ^{λ^*} which vanishes at $t=t_0$, t_1. The equations of variation can be written as

$$'\xi^{\lambda} = \xi^{\lambda} + \epsilon\omega^{\lambda}, \text{ conj.} \tag{3.4}$$

For clarity, we write

$$\phi(\xi^{\lambda}, \xi^{\lambda^*}, \dot{\xi}^{\lambda}, \dot{\xi}^{\lambda^*}) = (a_{\lambda\mu^*} d_t\xi^{\lambda} d_t\xi^{\mu^*})^{1/2}, \tag{3.5}$$

where

$$\dot{\xi}^{\lambda} = d_t\xi^{\lambda}, \qquad \dot{\xi}^{\lambda^*} = d_t\xi^{\lambda^*}. \tag{3.6}$$

Then by use of Taylor's series and integration by parts, we obtain for the first variation of (3.3)

[6] L. P. Eisenhart, loc. cit., p. 49.

$$(3.7)\quad 0 = \int_{t_0}^{t_1}\left[\frac{\partial\phi}{\partial\xi^\lambda} - \frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{\xi}^\lambda}\right)\right]\omega^\lambda dt + \left[\frac{\partial\phi}{\partial\xi^{\lambda^*}} - \frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{\xi}^{\lambda^*}}\right)\right]\omega^{\lambda^*}dt.$$

First, let us assume that the vector $(\omega^\lambda, \omega^{\lambda^*})$ has the components $(a, b, \cdots, a, b, \cdots)$ where $a, b, \cdots$ are arbitrary real functions of $\xi^\lambda, \xi^{\lambda^*}$. Secondly, let us assume that $(\omega^\lambda, \omega^{\lambda^*})$ has the components $(ia, ib, \cdots, -ia, -ib, \cdots)$. Then, by substituting into (3.7), we find that this relation decomposes into the two equations

$$(3.8)\quad 0 = \int_{t_0}^{t_1}\left[\frac{\partial\phi}{\partial\xi^\lambda} - \frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{\xi}^\lambda}\right)\right]\omega^\lambda dt,$$

$$(3.9)\quad 0 = \int_{t_0}^{t_1}\left[\frac{\partial\phi}{\partial\xi^{\lambda^*}} - \frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{\xi}^{\lambda^*}}\right)\right]\omega^{\lambda^*}dt.$$

By the ordinary argument, we obtain the Euler equations

$$(3.10)\quad \frac{\partial\phi}{\partial\xi^\lambda} - \frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{\xi}^\lambda}\right) = 0,$$

$$(3.11)\quad \frac{\partial\phi}{\partial\xi^{\lambda^*}} - \frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{\xi}^{\lambda^*}}\right) = 0.$$

We follow Eisenhart in listing the steps of our computation. From (3.5), we have

$$(3.12)\quad \frac{\partial\phi}{\partial\dot{\xi}^\lambda} = \frac{1}{2}\,\frac{a_{\lambda\mu^*}d_t\xi^{\mu^*}}{(a_{\lambda\mu^*}d_t\xi^\lambda d_t\xi^{\mu^*})^{1/2}} = \frac{1}{2}\,\frac{a_{\lambda\mu^*}d_t\xi^{\mu^*}}{(ds/dt)},$$

$$(3.13)\quad \frac{\partial\phi}{\partial\xi^\lambda} = \frac{1}{2}\,\frac{(\partial_\lambda a_{\nu\mu^*})d_t\xi^\nu d_t\xi^{\mu^*}}{(ds/dt)},$$

$$\frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{\xi}^\lambda}\right)$$

$$(3.14)\quad = \frac{1}{2}\,\frac{\dfrac{ds}{dt}[a_{\lambda\mu^*}d_t^2\xi^{\mu^*} + \partial_\nu a_{\lambda\mu^*}d_t\xi^\nu d_t\xi^{\mu^*} + \partial_\nu a_{\lambda\mu^*}d_t\xi^{\nu^*}d_t\xi^{\mu^*}] - \dfrac{d^2s}{dt^2}(a_{\lambda\mu^*}d_t\xi^{\mu^*})}{(ds/dt)^2}.$$

Substituting these relations into (3.5), we obtain

$$(3.15)\quad a_{\lambda\mu^*}d_t^2\xi^{\mu^*} + \partial_\nu a_{\lambda\mu^*}d_t\xi^\nu d_t\xi^{\mu^*} + \partial_\nu a_{\lambda\mu^*}d_t\xi^{\nu^*}d_t\xi^{\mu^*} - \partial_\lambda a_{\nu\mu^*}d_t\xi^\nu d_t\xi^{\mu^*} - \frac{a_{\lambda\mu^*}d_t\xi^{\mu^*}\dfrac{d^2s}{dt^2}}{(ds/dt)} = 0.$$

From (2.30) and (2.31), we find

$$\partial_\nu a_{\lambda\mu^*} = \Gamma^{\kappa}_{\nu\lambda} a_{\kappa\mu^*}, \tag{3.16}$$

$$\partial_{\nu^*} a_{\lambda\mu^*} = \Gamma^{\kappa^*}_{\nu^*\mu^*} a_{\lambda\kappa^*}. \tag{3.17}$$

Hence (3.15) becomes

$$a_{\lambda\mu^*} d_i^2\xi^{\mu^*} + a_{\kappa\mu^*}\Gamma^{\kappa}_{\nu\lambda} d_i\xi^{\nu} d_i\xi^{\mu^*} + \Gamma^{\kappa^*}_{\nu^*\mu^*} a_{\lambda\kappa^*} d_i\xi^{\nu^*} d_i\xi^{\mu^*} - \Gamma^{\kappa}_{\lambda\nu} a_{\kappa\mu^*} d_i\xi^{\nu} d_i\xi^{\mu^*}$$

$$- a_{\lambda\mu^*} d_i\xi^{\lambda^*} \frac{d^2s}{dt^2} \Big/ \frac{ds}{dt} = 0. \tag{3.18}$$

Transvecting (3.18) with $a^{\alpha^*\lambda}$, we obtain after combining terms two and four

$$d_i^2\xi^{\alpha^*} + \Gamma^{\alpha^*}_{\nu^*\mu^*} d_i\xi^{\nu^*} d_i\xi^{\mu^*} + 2a_{\kappa\mu^*} a^{\alpha^*\lambda} S^{\cdots\kappa}_{\nu\lambda}\, d_i\xi^{\nu} d_i\xi^{\mu^*} - d_i\xi^{\alpha^*}\left(\frac{d^2s}{dt^2}\Big/\frac{ds}{dt}\right) = 0. \tag{3.19}$$

From the equation (3.11), we obtain the conjugate of (3.19). Hence, we have this theorem:

THEOREM 1. *The geodesic X_1 in unitary K_n which are functions of a real parameter (t) is given by the solutions of the second order differential equation* (3.19) *and its conjugate.*

Evidently, the equations of the geodesic X_1 in unitary K_n differ from the equations of the geodesic V_1 in V_n in the additional term containing the torsion affinor. It follows that if the unitary space K_n has no torsion, then the geodesic X_1 of K_n satisfies the same type of differential equation as the geodesic V_1 in V_n and conversely.

4. **Unitary spaces with corresponding geodesics.** By multiplying the conjugate of (3.19) by $d_i\xi^{\beta}$, forming a similar equation with the indices α, β interchanged and then subtracting the two resulting equations, we obtain[7] for the geodesic X_1 in K_n

$$(d_i\xi^{\beta} d_i^2\xi^{\alpha} - d_i\xi^{\alpha} d_i^2\xi^{\beta}) + (\Gamma^{\alpha}_{\nu\mu} d_i\xi^{\beta} - \Gamma^{\beta}_{\nu\mu} d_i\xi^{\alpha}) d_i\xi^{\nu} d_i\xi^{\mu}$$

$$+ 2a_{\mu\kappa^*} S^{\cdots\kappa^*}_{\nu^*\lambda^*} (a^{\lambda^*\alpha} d_i\xi^{\beta} - a^{\lambda^*\beta} d_i\xi^{\alpha}) d_i\xi^{\nu^*} d_i\xi^{\mu} = 0, \text{ conj.} \tag{4.1}$$

Consider another n-dimensional unitary space $'K_n$ which is also mapped by the variables ξ^{λ} such that the point $P(\xi^{\lambda})$ of K_n corresponds to the point $'P(\xi^{\lambda})$ of $'K_n$. Furthermore, let the fundamental tensor and connection of this space be denoted by

$$'a_{\lambda\mu^*}, \qquad '\Gamma^{\kappa}_{\nu\mu}. \tag{4.2}$$

[7] L. P. Eisenhart, loc. cit., p. 131.

If the geodesics of ${}'K_n$ correspond to those of K_n, then equation (4.1) with $a_{\lambda\mu^*}$, $\Gamma^{\alpha}_{\lambda\mu}$, $S_{\nu\mu}^{\cdot\cdot\kappa}$ replaced by ${}'a_{\lambda\mu^*}$, ${}'\Gamma^{\alpha}_{\lambda\mu}$, ${}'S_{\nu\mu}^{\cdot\cdot\kappa}$ is satisfied by the same functions $\xi^{\alpha}(t)$ which satisfy (4.1), that is,

$$(d_t\xi^{\beta}d_t^2\xi^{\alpha} - d_t\xi^{\alpha}d_t^2\xi^{\beta}) + ({}'\Gamma^{\alpha}_{\nu\mu}d_t\xi^{\beta} - {}'\Gamma^{\beta}_{\nu\mu}d_t\xi^{\alpha})d_t\xi^{\nu}d_t\xi^{\mu} + 2\,{}'a_{\mu\kappa^*}\,{}'S_{\nu\lambda^*}^{\cdot\cdot\kappa^*}\,({}'a^{\lambda^*\alpha}d_t\xi^{\beta} - {}'a^{\lambda^*\beta}d_t\xi^{\alpha})d_t\xi^{\nu^*}d_t\xi^{\mu} = 0, \text{ conj.} \tag{4.3}$$

Let us first assume that K_n and ${}'K_n$ both possess symmetric connections.

THEOREM 2. *The necessary and sufficient condition that two unitary spaces K_n, ${}'K_n$ both with symmetric connections have geodesics in correspondence is that a vector p_ν exist such that*

$${}'\Gamma^{\alpha}_{\nu\mu} = \Gamma^{\alpha}_{\nu\mu} + 2p_{(\nu}A^{\alpha}_{\mu)}, \text{ conj.}$$

We can write the equation

$${}'\Gamma^{\alpha}_{\nu\mu} = \Gamma^{\alpha}_{\nu\mu} + A_{\nu\mu}^{\cdot\cdot\alpha}, \text{ conj.}, \tag{4.4}$$

where $A_{\nu\mu}^{\cdot\cdot\alpha}$ is an affinor to be determined. Subtracting (4.1) from (4.3) (note: the torsion affinors are zero in this case), we obtain

$$(A_{\nu\mu}^{\cdot\cdot\alpha}d_t\xi^{\beta} - A_{\nu\mu}^{\cdot\cdot\beta}d_t\xi^{\alpha})d_t\xi^{\nu}d_t\xi^{\mu} = 0, \text{ conj.} \tag{4.5}$$

By use of the unit affinor A^{α}_{γ}, we may rewrite (4.5) in the simpler form

$$(A_{\nu\mu}^{\cdot\cdot\alpha}A^{\beta}_{\gamma} - A_{\nu\mu}^{\cdot\cdot\beta}A^{\alpha}_{\gamma})d_t\xi^{\gamma}d_t\xi^{\nu}d_t\xi^{\mu} = 0, \text{ conj.} \tag{4.6}$$

Since (4.6) is to be satisfied by arbitrary values of $d_t\xi^{\alpha}$ at any point $P(\xi^{\lambda})$ and ${}'P(\xi^{\lambda})$, we find

$$A_{(\nu\mu}^{\cdot\cdot\alpha}A^{\beta}_{\gamma)} - A_{(\nu\mu}^{\cdot\cdot\beta}A^{\alpha}_{\gamma)} = 0, \text{ conj.} \tag{4.7}$$

Expanding (4.7), we obtain

$$A_{(\nu\mu)}^{\cdot\cdot\alpha}A^{\beta}_{\gamma} + A_{(\mu\gamma)}^{\cdot\cdot\alpha}A^{\beta}_{\nu} + A_{(\nu\gamma)}^{\cdot\cdot\alpha}A^{\beta}_{\mu} = A_{(\nu\mu)}^{\cdot\cdot\beta}A^{\alpha}_{\gamma} + A_{(\mu\gamma)}^{\cdot\cdot\beta}A^{\alpha}_{\nu} + A_{(\nu\gamma)}^{\cdot\cdot\beta}A^{\alpha}_{\mu}, \text{ conj.} \tag{4.8}$$

Let us write

$$A_{\nu\alpha}^{\cdot\cdot\alpha} = A_{\alpha\nu}^{\cdot\cdot\alpha} = (n+1)p_{\nu}, \text{ conj.}, \tag{4.9}$$

and contract on the indices μ, α in (4.8). Then we obtain

$$A_{\nu\gamma}^{\cdot\cdot\beta} = 2p_{(\nu}A^{\beta}_{\gamma)}, \text{ conj.} \tag{4.10}$$

Conversely, if (4.10) is satisfied, then (4.6) is identically satisfied. Hence our theorem is proved.

Let us now assume that K_n has torsion and $'K_n$ has no torsion (symmetric connection).

THEOREM 3. *Two unitary spaces, K_n with torsion, $'K_n$ without torsion, cannot have their geodesics in correspondence.*

Let us assume the contrary, namely, that the geodesics of these two spaces correspond. Then again, we write the equation (4.4) and again, we subtract (4.1) from (4.3). In this case, instead of equation (4.6), we obtain

$$(4.11)\qquad \begin{aligned}&(A_{\nu\mu}^{\cdot\cdot\alpha}A_{\gamma}^{\beta}-A_{\nu\mu}^{\cdot\cdot\beta}A_{\gamma}^{\alpha})d_i\xi^{\gamma}d_i\xi^{\nu}d_i\xi^{\mu}\\&\quad+2a_{\mu\kappa^*}S_{\cdot\nu\lambda^*}^{\cdot\cdot\kappa^*}(a^{\lambda^*\alpha}A_{\gamma}^{\beta}-a^{\lambda^*\beta}A_{\gamma}^{\alpha})d_i\xi^{\gamma}d_i\xi^{\mu}d_i\xi^{\nu^*}=0,\ \text{conj.}\end{aligned}$$

Equation (4.11) is to be satisfied by arbitrary values of the $d_i\xi^{\alpha}$. Let us first assume that the vector $(d_i\xi^{\alpha}, d_i\xi^{\alpha^*})$ has the arbitrary real components $(a, b, c, \cdots a, b, c, \cdots)$ and secondly, let us assume that this vector has the arbitrary pure imaginary components $(ia, ib, ic, \cdots -ia, -ib, \cdots)$. By adding the resulting equations, we find that (4.11) decomposes into the two equations

$$(4.12)\qquad (A_{\nu\mu}^{\cdot\cdot\alpha}A_{\gamma}^{\beta}-A_{\nu\mu}^{\cdot\cdot\beta}A_{\gamma}^{\alpha})d_i\xi^{\gamma}d_i\xi^{\nu}d_i\xi^{\mu}=0,\ \text{conj.},$$

$$(4.13)\qquad a_{\mu\kappa^*}S_{\cdot\nu\lambda^*}^{\cdot\cdot\kappa^*}(a^{\lambda^*\alpha}A_{\gamma}^{\beta}-a^{\lambda^*\beta}A_{\gamma}^{\alpha})d_i\xi^{\gamma}d_i\xi^{\mu}d_i\xi^{\nu^*}=0,\ \text{conj.}$$

The equation (4.12) can be analyzed in an exactly analogous manner to that used in the previous theorem excepting that (4.9) must be replaced by

$$(4.14)\qquad A_{\nu\alpha}^{\cdot\cdot\alpha}=(n+1)p_{\nu},\qquad A_{\alpha\nu}^{\cdot\cdot\alpha}=(n+1)q_{\nu},\ \text{conj.},$$

since $A_{\nu\mu}^{\cdot\cdot\alpha}$ is no longer symmetric in ν, μ. Instead of (4.10), we find

$$(4.15)\qquad A_{(\nu\gamma)}^{\cdot\cdot\beta}=2r_{(\nu}A_{\gamma)}^{\beta},\qquad 2r_{\nu}=p_{\nu}+q_{\nu}\ \text{conj.}$$

In order to analyze (4.13), we write this equation in the form

$$(4.16)\qquad A_{\mu\gamma\nu^*}^{\alpha\beta}d_i\xi^{\mu}d_i\xi^{\gamma}d_i\xi^{\nu^*}=0,\ \text{conj.},$$

where

$$(4.17)\qquad A_{\mu\gamma\nu^*}^{\alpha\beta}=a_{\mu\kappa^*}S_{\cdot\nu\lambda^*}^{\cdot\cdot\kappa}(a^{\lambda^*\alpha}A_{\gamma}^{\beta}-a^{\lambda^*\beta}A_{\gamma}^{\alpha}),\ \text{conj.}$$

As previously, let us assume first that $(d_i\xi^{\alpha}, d_i\xi^{\alpha^*})$ has the arbitrary real components $(a, b, \cdots a, b, \cdots)$. Upon substituting into (4.16), we obtain a third degree homogeneous polynomial. We shall explicitly

write those terms which contain a, b; the remaining terms are formulated in a similar fashion. We have

$$A^{\alpha\beta}_{111\cdot}a^3 + a^2b(A^{\alpha\beta}_{112\cdot} + A^{\alpha\beta}_{121\cdot} + A^{\alpha\beta}_{211\cdot}) + ab^2(A^{\alpha\beta}_{122\cdot} + A^{\alpha\beta}_{212\cdot} + A^{\alpha\beta}_{221\cdot}) + A^{\alpha\beta}_{222\cdot}b^3 + \cdots = 0. \tag{4.18}$$

Secondly, let us assume $(d_t\xi^\alpha, d_t\xi^{\alpha^*})$ has the arbitrary components $(a, ib, \cdots a, ib, \cdots)$. Then (4.16) becomes

$$A^{\alpha\beta}_{111\cdot}a^3 + a^2bi(-A^{\alpha\beta}_{112\cdot} + A^{\alpha\beta}_{121\cdot} + A^{\alpha\beta}_{211\cdot}) + ab^2(A^{\alpha\beta}_{122\cdot} + A^{\alpha\beta}_{212\cdot} - A^{\alpha\beta}_{221\cdot}) - iA^{\alpha\beta}_{222\cdot}b^3 + \cdots = 0. \tag{4.19}$$

From these equations, that is, (4.18), (4.19) we conclude

$$A^{\alpha\beta}_{111\cdot} = A^{\alpha\beta}_{222\cdot} = A^{\alpha\beta}_{122\cdot} = A^{\alpha\beta}_{221\cdot} = 0, \quad A^{\alpha\beta}_{(12)1\cdot} = A^{\alpha\beta}_{(12)2\cdot} = 0. \tag{4.20}$$

We can evidently express (4.20) by writing

$$A^{\alpha\beta}_{(\mu\gamma)\nu\cdot} = 0, \text{ conj.} \tag{4.21}$$

By use of (4.17), we find upon expanding

$$S_{\nu\cdot\lambda^*}^{\cdots\varepsilon^*}(a_{\mu\kappa\cdot}a^{\lambda^*\alpha}A^{\beta}_{\gamma} + a_{\gamma\kappa\cdot}a^{\lambda^*\alpha}A^{\beta}_{\mu} - a_{\mu\kappa\cdot}a^{\lambda^*\beta}A^{\alpha}_{\gamma} - a_{\gamma\kappa\cdot}a^{\lambda^*\beta}A^{\alpha}_{\mu}) = 0, \text{ conj.} \tag{4.22}$$

Placing α equal to μ and contracting, we find

$$S_{\nu\cdot\lambda^*}^{\cdots\lambda^*}A^{\beta}_{\gamma} = nS_{\nu\cdot\lambda^*}^{\cdots\varepsilon^*}a_{\gamma\kappa\cdot}a^{\lambda^*\beta}, \text{ conj.} \tag{4.23}$$

Solving (4.23) for $S_{\nu\cdot\lambda^*}^{\cdots\varepsilon^*}$, we find

$$S_{\nu\cdot\mu^*}^{\cdots\alpha^*} = S_{\nu\cdot}A_{\mu^*}^{\alpha^*}, \text{ conj.}, \tag{4.24}$$

where

$$S_{\nu\cdot\lambda^*}^{\cdots\lambda^*} = nS_{\nu\cdot}, \text{ conj.} \tag{4.25}$$

Since $S_{\nu\cdot\lambda^*}^{\cdots\alpha^*}$ is antisymmetric in ν^*, μ^* we have

$$S_{(\nu\cdot}A_{\mu^*)}^{\alpha^*} = 0, \text{ conj.} \tag{4.26}$$

Contracting on α^*, μ^* we obtain the relation

$$S_{\nu\cdot} = 0, \text{ conj.} \tag{4.27}$$

Hence $S_{\nu\cdot\mu^*}^{\cdots\alpha^*}$ vanishes. But this is contrary to our assumption. Hence our theorem is proved.

UNIVERSITY OF TEXAS

SUR LES SOLUTIONS DES ÉQUATIONS DU TYPE PARABOLIQUE DÉTERMINÉES DANS UNE RÉGION ILLIMITÉE

MIROSLAW KRZYŻAŃSKI

L'un des problèmes rélatifs à l'équation de la chaleur consiste en la recherche d'une solution de cette équation, se réduisant à une fonction donnée sur la caractéristique. Cette solution est donnée par l'intégrale classique de Weierstrass.[1] Cependant la question de l'unicité de cette solution n'a été complètement résolue que dans les travaux récentes de M. M. Picone et M. A. Tychonoff.[2]

Or il y a intérêt à éteindre ces résultats à l'équation linéaire du type parabolique:

$$\sum_{i,k=1}^{m} A_{ik}(x_1, x_2, \cdots, x_m, y)\frac{\partial^2 u}{\partial x_i \partial x_k} = \sum_{j=1}^{m} a_j(x_1, x_2, \cdots, x_m, y)\frac{\partial u}{\partial x_j} + b(x_1, x_2, \cdots, x_m, y)\frac{\partial u}{\partial y} + c(x_1, x_2, \cdots, x_m, y)u, \tag{1}$$

la forme $\sum_{i,k=1}^{m} A_{ik}\lambda_i\lambda_k$ étant positive définie, en cherchant la solution u de cette équation déterminée dans une région illimitée R: $0 \leqq y \leqq h$; $-\infty < x_i < +\infty$ $(i=1, 2, \cdots, m)$, telle que

$$u(x_1, x_2, \cdots, x_m, 0) = \phi(x_1, x_2, \cdots, x_m).$$

Nous commençons par introduire une classification des fonctions caractérisant la façon dont elles se comportent à l'infini. Conviendrons d'appeler dans la suite la classe E_α $(\alpha > 0)$ une classe de fonctions $F(x)$ qui ont la propriété suivante: il existe deux nombres constants positifs M et K tels que

$$|F(x)| < Me^{K|x|^\alpha}, \qquad -\infty < x < +\infty.$$

Nous allons voir que dans la classe E_2 par rapport aux x_i il existe une solution et une seule du problème posé.

Le résultat analogue pour l'équation de la chaleur a été obtenu par

[1] Voir par exemple E. Goursat, *Cours d'Analyse*, Tome 3, Paris, 1927, chap. 29, p. 294.

[2] P. Picone, *Sul problema della propagazione del calore in un mezzo privo di frontiera*, Mathematische Annalen, Tome 101 (1929), pp. 701–712. A. Tychonoff, *Théorèmes d'unicité pour l'équation de chaleur*, Recueil Mathématique de Moscou, Tome 42 (1935), pp. 199–214. Cf. aussi le travail récent de M. F. G. Dressel, *The fundamental solution of the parabolic equation*, Duke Mathematical Journal, Tome 7 (1940), pp. 186–203.

M. Tychonoff (voir le travail cité), qui a démontré en outre que l'unicité du problème n'a pas lieu dans la classe $E_{2+\epsilon}$, où ϵ est un nombre positif arbitrairement petit.

Considérons l'équation de la forme (1) dont les coefficients sont continus et bornés dans la région R; plus précisemment, supposons l'existence des nombres positifs A, α et γ tels que: $|A_{ik}| < A$, $|a_j| < \alpha$, $|c| < \gamma$, $(i, k, j = 1, 2, \cdots, m)$, dans toute la région R; quant au coefficient b, on suppose l'existence des nombres β et β_1 tels que $\beta_1 > b > \beta > 0$ partout dans R. Tout ceci étant supposé, nous allons démontrer le théorème suivant:

THÉORÈME I. *La seule solution de l'équation* (1) *appartenant à la classe* E_2 *par rapport aux* x_i *et s'annulant sur la caractéristique* $y = 0$ *est* $u \equiv 0$.

DÉMONSTRATION. D'après la règle générale due à M. Picone (voir le travail cité) on transforme l'équation (1) en posant $u(x_1, x_2, \cdots, x_m, y) = v(x_1, x_2, \cdots, x_m, y) \cdot W(x_1, x_2, \cdots, x_m, y)$, v étant la nouvelle fonction inconnue; W étant supposé différent de zéro, on obtient ainsi

$$\sum_{i,k=1}^{m} A_{i,k} \frac{\partial^2 v}{\partial x_i \partial x_k} = \sum_{j=1}^{m} a_j^{(1)} \frac{\partial v}{\partial x_j} + b \frac{\partial v}{\partial y} + c^{(1)} v \tag{2}$$

et on doit choisir W de sorte que 1° le coefficient $c^{(1)}$ soit positif dans R, 2° $v \to 0$ pour $\sum_{i=1}^{m} x_i^2 \to \infty$.

Passons donc à la construction de la fonction W conforme à notre problème. La fonction u appartenant à la classe E_2, il existe deux nombres M et K_0 tels que

$$|u(x_1, x_2, \cdots, x_m, y)| < M \exp\left(K_0 \cdot \sum_{i=1}^{m} x_i^2\right) \tag{3}$$

dans R. Soit

$$\mu = \frac{5K}{\beta}; \qquad \nu = \frac{1}{\beta}\left[m\left(1 + \frac{\alpha}{A}\right)^2 + K + \gamma\right]$$

où $K > K_0$ et posons

$$u(x_1, x_2, \cdots, x_m, y) = v(x_1, x_2, \cdots, x_m, y) \cdot H(x_1, x_2, \cdots, x_m, y; K),$$

avec

$$H(x_1, x_2, \cdots, x_m, y; K) = \exp\left[\frac{K \sum_{i=1}^{m} x_i^2}{1 - \mu y} + \nu y\right]. \tag{4}$$

Supposons que la hauteur h de R est inférieure à $1/\mu$.

Ainsi dans la transformation de M. Picone nous avons remplacé la fonction W par H. Un calcul montre qu'en chaque point de R le coefficient $c^{(1)}$ de l'équation (2) est supérieur à une forme quadratique en x_i positive définie, à savoir:

$$c^{(1)} > A\sum_{j=1}^{m}\left[\frac{K|x_j|}{1-\mu y}-\left(1+\frac{\alpha}{A}\right)\right]^2+K.$$

Il en résulte que si l'on construit un parallélipipède rectangulaire R_n, détaché de R par les plans $x_i=\pm n$, *la fonction v ne pourrait atteindre dans* R_n *ni un maximum positif, ni un minimum négatif que sur la partie de sa surface située sur les plans* $x_i=\pm n$ *et* $y=0$ (voir le travail cité de M. Picone).

Soit $P_0(x_1^{(0)}, x_2^{(0)}, \cdots, x_m^{(0)}, y^{(0)})$ un point quelconque de R; pour n assez grand R_n contiendra P_0. En vertu de (3) on peut déterminer n de sorte que $|v|<\epsilon$ sur la surface latérale de R_n, $\epsilon>0$ étant un nombre arbitrairement petit. D'autre part v s'annule sur le plan $y=0$, donc $|v(x_1^{(0)}, x_2^{(0)}, \cdots, x_m^{(0)}, y^{(0)})|<\epsilon$. Il en résulte que $v(x_1^{(0)}, x_2^{(0)}, \cdots, x_m^{(0)}, y^{(0)})=0$ et, par suite, $u(x_1^{(0)}, x_2^{(0)}, \cdots, x_m^{(0)}, y^{(0)})=0$. Comme P_0 est un point arbitraire de R, on a

$$u(x_1, x_2, \cdots, x_m, y) \equiv 0$$

dans R.

Passons à la démonstration de l'existence de la solution du problème, en supposant que les *coefficients* $A_{i,k}$ *et* b *admettent les dérivées du premier ordre continues.*

THÉORÈME II. *La fonction* $\phi(x_1, x_2, \cdots, x_m)$ *étant de la classe* E_2 *et continue, il existe une solution et une seule de l'équation* (1) *appartenant à la classe* E_2 *en tant que la fonction des* x_i *déterminée dans la région R de hauteur h (suffisamment petite) et se réduisant à la fonction* ϕ *sur la caractéristique* $y=0$.

DÉMONSTRATION. L'unicité de cette solution résulte aussitôt du Théorème I. Il suffit donc de démontrer son existence.

La fonction ϕ étant de la classe E_2, il existe deux nombres M et K tels que

$$|\phi(x_1, x_2, \cdots, x_m)| < M\exp\left(K\sum_{i=1}^{m}x_i^2\right). \tag{5}$$

Construisons la parallélipipède R_p, analogue à R_n (voir la démonstration du Théorème I) et soit S_p sa surface latérale, composé des parties des plans $x_i=\pm p$. D'après les résultats de M. Giraud et

M. Gevrey[3] il existe une solution u_p de (1) telle que:

$$u_p(x_1, x_2, \cdots, x_m, 0) = \phi(x_1, x_2, \cdots, x_m),$$
$$u_p(x_1, x_2, \cdots, x_m, y) = \phi(x_1, x_2, \cdots, x_m) \qquad \text{sur } S_p.$$

Cette solution est déterminée dans R_p. D'une manière analogue définissons la solution u_q de (1), déterminée dans le parallélipipède R_q $(q>p)$ analogue à R_p; soit S_q sa surface latérale.

Supposons que $h<2/KA$ et soient v_p, v_q et v_q^* les fonctions définies par les égalités

$$u_p = v_p \cdot H(x_1, x_2, \cdots, x_m, y; 2K);\ u_q = v_q \cdot H(x_1, x_2, \cdots, x_m, y; 2K);$$
$$u_q = v_q^* \cdot H(x_1, x_2, \cdots, x_m, y; K).$$

Observons que $H(x_1, x_2, \cdots, x_m, y; 2K) > H(x_1, x_2, \cdots, x_m, y; K) \cdot \exp(\overline{K}\sum_{i=1}^{m} x_i^2)$ ($\overline{K}$ étant un nombre positif) de sorte que

$$v_q < v_q^* \exp\left(-\overline{K}\sum_{i=1}^{m} x_i^2\right); \tag{6}$$

on peut supposer que $\overline{K}<K$. D'après (5) et la définition de H on a $|v_q^*|<M$ sur S_q et $|v_q^*(x_1, x_2, \cdots, x_m, 0)|<M$; il en résulte que $|v_q^*|<M$ partout dans R_q et en particulier sur S_p. En égard à (6) on a donc $|v_q|<Me^{-\overline{K}mp^2}$ sur S_p.

La fonction:

$$w_{pq}(x_1, x_2, \cdots, x_m, y) = v_q(x_1, x_2, \cdots, x_m, y) - v_p(x_1, x_2, \cdots, x_m, y)$$

est déterminée partout dans R_p et l'on a $|w_{pq}|<2M\exp(-\overline{K}mp^2)$ sur S_p et $w_{pq}(x_1, x_2, \cdots, x_m, 0)=0$. Donc

$$|w_{pq}(x_1, x_2, \cdots, x_m, y)| < 2Me^{-\overline{K}mp^2} \qquad \text{partout dans } R_p. \tag{7}$$

Soit maintenant P_0 un point quelconque de R; considérons un parallélipipède fixe $\rho \subset R$ de hauteur h, entourant P_0. D'après (7) on peut déterminer un nombre p_0 de sorte que l'on ait

$$|u_q(x_1, x_2, \cdots, x_m, y) - u_p(x_1, x_2, \cdots, x_m, y)| < \epsilon$$

pour $q>p>p_0$ partout dans ρ. Ceci montre que la suite $\{u_p\}$ tend vers une limite $U(x_1, x_2, \cdots, x_m, y)$ partout dans R, le point P_0 étant un point arbitraire de R; en particulier $U_p \to U$ uniformement dans ρ.

Nous démontrerons que U est précisemment la solution de notre

[3] G. Giraud, *Sur certaines opérations aux dérivées partielles du type parabolique*, Comptes Rendus de l'Académie des Sciences, Paris, Tome 195 (1932), pp. 98–100. M. Gevrey, *Systèmes d'équations aux dérivées partielles du type parabolique*, ibid., Tome 195 (1932), pp. 690–693.

problème. On a évidemment $U(x_1, x_2, \cdots, x_m, y) = \phi(x_1, x_2, \cdots, x_m)$; il nous reste à prouver que U satisfait à l'équation (1) partout dans R. Il suffit évidemment de démontrer que ceci a lieu au point P. Or soit $\bar{u}(x_1, x_2, \cdots, x_m, y)$ la solution de (1) déterminée dans ρ, égale à ϕ pour $y=0$ et identique à U sur la surface latérale σ de ρ. Nous allons voir que $\bar{u} \equiv U$ dans ρ.

En effet, $\epsilon > 0$ étant un nombre arbitrairement petit, on peut déterminer le nombre p de sorte que

$$\left| u_p(x_1^{(0)}, x_2^{(0)}, \cdots, x_m^{(0)}, y^{(0)}) - U(x_1^{(0)}, x_2^{(0)}, \cdots, x_m^{(0)}, y^{(0)}) \right| < \epsilon/2,$$

$$\left| u_p(x_1, x_2, \cdots, x_m, y) - U(x_1, x_2, \cdots, x_m, y) \right| < \epsilon/2 \qquad \text{sur } \sigma.$$

Il résulte de la dernière inégalité, que

$$\left| u_p(x_1^{(0)}, x_2^{(0)}, \cdots, x_m^{(0)}, y^{(0)}) - \bar{u}(x_1^{(0)}, x_2^{(0)}, \cdots, x_m^{(0)}, y^{(0)}) \right| < \epsilon/2$$

puisque $u_p - \bar{u}$ est dans ρ une solution de (1) et on a $u_p = \bar{u}$ pour $y=0$ et $\bar{u} \equiv U$ sur σ. On en déduit $|\bar{u}(x_1^{(0)}, x_2^{(0)}, \cdots, x_m^{(0)}, y^{(0)}) - U(x_1^{(0)}, x_2^{(0)}, \cdots, x_m^{(0)}, y^{(0)})| < \epsilon$ donc

$$\bar{u}(x_1^{(0)}, x_2^{(0)}, \cdots, x_m^{(0)}, y^{(0)}) = U(x_1^{(0)}, x_2^{(0)}, \cdots, x_m^{(0)}, y^{(0)}).$$

Ainsi U est une intégrale de (1) dans R, c'est donc la solution de notre problème.

Il nous reste encore à démontrer qu'elle est de la classe E_2 par rapport aux x_i. Or la fonction $v^* = \lim_{p\to\infty} v_p^*$ satisfait à l'inégalité $|v^*| < M$. Comme d'autre part $U = v^* \cdot H(x_1, x_2, \cdots, x_m, y; K)$, on a

$$\left| U(x_1, x_2, \cdots, x_m, y) \right| < M \exp\left[\frac{K \sum_{i=1}^{m} x_i^2}{1 - \mu y} + \nu y \right].$$

Si la fonction ϕ est de la classe E_1, on démontrerait aisément, qu'il en est de même de U (par rapport aux x_i); plus précisemment, l'inégalité $|\phi| < M \exp\left[K(\sum_{i=1}^{m} x_i^2)^{1/2}\right]$ entraine

$$|U| < M \exp K\left[\left(\sum_{i=1}^{m} x_i^2\right)^{1/2} + \Gamma(y)\right],$$

Γ étant une fonction bornée. En particulier l'inégalité $|\phi| < M$ entraine $|U| < M$ dans R.

WILNO, POLAND

ON APPROXIMATION BY EUCLIDEAN AND NON-EUCLIDEAN TRANSLATIONS OF AN ANALYTIC FUNCTION

W. SEIDEL AND J. L. WALSH

In 1929 G. D. Birkhoff established[1] the noteworthy result that an entire function $F(z)$ exists such that to an arbitrary entire function $g(z)$ corresponds a sequence $a_1, a_2, \cdots$ depending on $g(z)$ with the property

$$\lim_{n\to\infty} F(z + a_n) = g(z) \tag{1}$$

for all z, uniformly for z on every closed bounded set.

It is the object of the present note (a) to indicate that not merely an arbitrary entire function $g(z)$ can be expressed in the form (1), but also any function analytic in a simply connected region, and (b) to study the non-euclidean analogue of the entire problem; precisely analogous results are obtained. Some related topics under (a) have recently been studied by A. Roth,[2] who, however, does not mention the results to be proved here.

The immediate occasion of the interest of the present writers[3] in the problem is through (b), for non-euclidean translations have been widely used in the study of derivatives of univalent and other functions analytic in the unit circle $|z| = 1$; limit functions under such translations are of great significance in the study of derivatives and of limit values of a given function as a variable point z approaches the circumference $|z| = 1$.

We shall give a proof of the following theorem, proof and theorem differing only in detail from those of Birkhoff:

THEOREM 1. *There exists an entire function $F(z)$ such that given an arbitrary function $f(z)$ analytic in a simply connected region R of the z-plane, we have for suitably chosen $a_1, a_2, \cdots$ the relation*

$$\lim_{n\to\infty} F(z + a_n) = f(z) \tag{2}$$

for z in R, uniformly on any closed bounded set in R.

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[1] Comptes Rendus de l'Académie des Sciences, Paris, vol. 189, pp. 473–475.

[2] Comentarii Mathematici Helvetici, vol. 11 (1938–1939), pp. 77–125.

[3] Compare Seidel and Walsh, *On the derivatives of functions analytic in the unit circle and their radii of univalence and of p-valence*, a forthcoming paper in the Transactions of this Society.

Here and throughout the sequel we are concerned with the finite plane, that is to say, the plane of finite points z.

By way of geometric entities, we introduce the circles C_1: $|z-4| = 2$, C_2: $|z-4^2| = 2^2, \cdots$, C_n: $|z-4^n| = 2^n, \cdots$, and also the circles Γ_n: $|z| = 4^n+2^n+1$; it follows that the C_n are mutually exterior, and that Γ_n contains in its interior all the circles $C_1, C_2, \cdots, C_n$ but no point in or on any of the circles $C_{n+1}, C_{n+2}, \cdots$.

Let us enumerate the polynomials in z with rational coefficients: $p_1(z), p_2(z), \cdots$. It is of course true that any sequence of polynomials can be replaced by a sequence of polynomials with rational coefficients, without altering whatever properties may exist of convergence or uniform convergence to a given function on bounded point sets.

We choose $\pi_1(z)$ as a polynomial in z which satisfies the inequality $|p_1(z-4)-\pi_1(z)| < 1/2$, z on or within C_1; indeed we may choose $\pi_1(z) \equiv p_1(z-4)$. We choose $\pi_2(z)$ as a polynomial in z which satisfies the two inequalities

$$|\pi_1(z) - \pi_2(z)| < 1/4, \qquad z \text{ on or within } \Gamma_1,$$
$$|p_2(z-4^2) - \pi_2(z)| < 1/4, \qquad z \text{ on or within } C_2;$$

such a polynomial $\pi_2(z)$ exists, by Runge's classical theorem. In general, let $\pi_n(z)$ be a polynomial in z which satisfies the inequalities

$$|\pi_{n-1}(z) - \pi_n(z)| < 1/2^n, \quad z \text{ on or within } \Gamma_{n-1},$$
$$|p_n(z-4^n) - \pi_n(z)| < 1/2^n, \quad z \text{ on or within } C_n.$$

The sequence $\{\pi_n(z)\}$ converges uniformly in each of the circles Γ_m; hence converges at every point of the plane, uniformly on any bounded set. The limit function $F(z)$ is entire, and has the required properties. Indeed, let $f(z)$ be analytic in a simply connected region R; there exist polynomials $p_{n_k}(z)$ of the set already defined with

$$\lim_{n_k \to \infty} p_{n_k}(z) = f(z) \tag{3}$$

at every point of R, uniformly on any closed bounded set in R. For z in C_n: $|z-4^n| < 2^n$ we have

$$\begin{aligned} F(z) &= \pi_n(z) + [\pi_{n+1}(z) - \pi_n(z)] \\ &\quad + [\pi_{n+2}(z) - \pi_{n+1}(z)] + \cdots, \\ |F(z) - p_n(z-4^n)| &\leqq |p_n(z-4^n) - \pi_n(z)| + |\pi_{n+1}(z) - \pi_n(z)| \\ &\quad + |\pi_{n+2}(z) - \pi_{n+1}(z)| + \cdots \\ &< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots = \frac{1}{2^{n-1}}, \end{aligned}$$

whence

$$\lim_{n\to\infty} [F(z+4^n) - p_n(z)] = 0 \tag{4}$$

for every z, uniformly on any bounded set. To return to $f(z)$, we now have from (3) and (4)

$$\lim_{n_k\to\infty} [F(z+4^{n_k}) - f(z)] = 0 \tag{5}$$

for z in R, uniformly on any closed bounded set in R. Theorem 1 is established.

The special case of Theorem 1 that $f(z)$ is an entire function and R is the (finite) z-plane is included here, and is the case considered by Birkhoff. We add the remark that whenever a function $g(z)$ can be represented on a point set E (bounded or unbounded) by a sequence of polynomials, that function can also be represented on E in the manner indicated by (2), with preservation of the property of uniform convergence whenever that occurs on a bounded set belonging to E. For instance, E may consist of a sequence of disjoint simply connected regions $R_1, R_2, \cdots$, with $g(z)$ analytic on E; then $g(z)$ can be represented on E either by a sequence of polynomials or, as in (2), with uniform convergence on any closed bounded subset of E. On the general subject of representation by polynomials there exist modern researches due to Montel, Walsh, Hartogs and Rosenthal, and Lavrentieff.[4]

A further remark in connection with Theorem 1 is that *if the numbers* $A_0, A_1, A_2, \cdots$ *are arbitrary, there exists a sequence* $a_1, a_2, \cdots$ *with the property*

$$\lim_{n\to\infty} F^{(k)}(a_n) = A_k, \qquad k = 0, 1, 2, \cdots. \tag{6}$$

To establish (6) it is sufficient to remark that when m is given, the number a_m exists with the property

$$\left| F(z+a_m) - \left[A_0 + A_1 z + \frac{A_2}{2!} z^2 + \cdots + \frac{A_m}{m!} z^m \right] \right| < \frac{1}{2^m \cdot m!}, \qquad \text{for } |z| \leqq 1;$$

from Cauchy's inequality it then follows that we have $|F^{(k)}(a_m) - A_k| < 1/2^m$, $k = 0, 1, 2, \cdots, m$; the relation (6) follows.

[4] The reader may refer to Lavrentieff, *Sur les Fonctions d'une Variable Complexe Représentables par des Séries de Polynomes*, Actualités Scientifiques et Industrielles, no. 441, Paris, 1936.

We turn now to the non-euclidean analogue of Theorem 1:

THEOREM 2. *There exists a function $\Phi(z)$ analytic in the region $|z|<1$ such that given an arbitrary function $\phi(z)$ analytic in a simply connected subregion R, we have for suitably chosen $\alpha_1, \alpha_2, \cdots$ the relation*

$$\lim_{n\to\infty} \Phi\left(\frac{z+\alpha_n}{1+\bar{\alpha}_n z}\right) = \phi(z), \tag{7}$$

for z in R, uniformly on any closed set interior to R.

As in Theorem 1 we needed to use only real a_n, so here we shall actually employ only real α_n.

For geometric entities we choose here C_1 as the n.e. circle of n.e. radius 2 whose n.e. center is the point $z=\beta_1$ of the axis of reals whose n.e. distance from $z=0$ is 4, and in general choose C_n as the n.e. circle of n.e. radius 2^n whose n.e. center is the point $z=\beta_n$ of the axis of reals whose n.e. distance from $z=0$ is 4^n. Let Γ_n be the circle whose center is $z=0$ and n.e. radius 4^n+2^n+1, so that Γ_n contains in its interior all the circles $C_1, C_2, \cdots, C_n$, but no point in or on any of the circles $C_{n+1}, C_{n+2}, \cdots$.

As before, we use the polynomials $p_1(z), p_2(z), \cdots$ with rational coefficients. Choose $\pi_1(z)$ as a polynomial in z which satisfies the inequality

$$\left| p_1\left(\frac{z-\beta_1}{1-\beta_1 z}\right) - \pi_1(z) \right| < 1/2, \qquad z \text{ on or within } C_1;$$

choose $\pi_2(z)$ as a polynomial in z which satisfies the two inequalities

$$|\pi_1(z) - \pi_2(z)| < 1/4, \qquad z \text{ on or within } \Gamma_1,$$

$$\left| p_2\left(\frac{z-\beta_2}{1-\beta_2 z}\right) - \pi_2(z) \right| < 1/4, \qquad z \text{ on or within } C_2.$$

In general, let $\pi_n(z)$ be a polynomial in z which satisfies

$$|\pi_{n-1}(z) - \pi_n(z)| < 1/2^n, \quad z \text{ on or within } \Gamma_{n-1};$$

$$\left| p_n\left(\frac{z-\beta_n}{1-\beta_n z}\right) - \pi_n(z) \right| < 1/2^n, \qquad z \text{ on or within } C_n.$$

The sequence $\{\pi_n(z)\}$ converges uniformly in each of the circles Γ_m, hence converges at every point of the region $|z|<1$, uniformly on any closed subset. The limit function $\Phi(z)$ is analytic throughout the region $|z|<1$, and will now be shown to have the required properties.

For z in C_n we have

$$\Phi(z) = \pi_n(z) + [\pi_{n+1}(z) - \pi_n(z)] + [\pi_{n+2}(z) - \pi_{n+1}(z)] + \cdots,$$

$$\left|\Phi(z) - p_n\left(\frac{z-\beta_n}{1-\beta_n z}\right)\right| \leqq \left|p_n\left(\frac{z-\beta_n}{1-\beta_n z}\right) - \pi_n(z)\right| + |\pi_{n+1}(z) - \pi_n(z)| + \cdots$$

$$< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots = \frac{1}{2^{n-1}},$$

whence

$$\lim_{n\to\infty}\left[\Phi\left(\frac{z+\beta_n}{1+\beta_n z}\right) - p_n(z)\right] = 0 \tag{8}$$

for every z in $|z|<1$, uniformly on any closed set in $|z|<1$.

Let $\phi(z)$ be analytic in the simply connected region R in $|z|<1$. There exist polynomials $p_{n_k}(z)$ of the set already defined with

$$\lim_{n_k\to\infty} p_{n_k}(z) = \phi(z) \tag{9}$$

at every point of R, uniformly on any closed subset. From (8) and (9) we find

$$\lim_{n_k\to\infty}\left[\Phi\left(\frac{z+\beta_{n_k}}{1+\beta_{n_k} z}\right) - \phi(z)\right] = 0 \tag{10}$$

for z in R, uniformly on any closed set in R.

Theorem 2 is established. The region R may in particular be the region $|z|<1$. Remarks for Theorem 2 entirely analogous to those for Theorem 1 are also valid.

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ZERO-DIMENSIONAL FAMILIES OF SETS[1]

SAMUEL EILENBERG AND E. W. MILLER

A family $\Phi=\{A_\alpha\}$ of subsets of a topological space X will be called 0-*dimensional* if given an open set U such that $A_{\alpha_0}\subset U$, there is an open set V such that (1) $A_{\alpha_0}\subset V\subset U$ and (2) $(\overline{V}-V)\sum_\alpha A_\alpha=0$. We enumerate below a few of the most common 0-dimensional families. In each case the proof of 0-dimensionality is easy, and is therefore omitted.

(I) Every family of disjoint open subsets of a topological space is 0-dimensional.

(II) Let Y be a locally connected subset of a topological space X. The family Φ of the components of Y is 0-dimensional.

(III) Let Y be a compact and closed subset of a metric space X. The family Φ of the components of Y is 0-dimensional.

(IV) Let Y be a subset of a metric space X. The family Φ consisting of the individual points of Y is 0-dimensional if and only if dim $Y=0$.

(V) Let Φ be a family of closed subsets of a compact metric space X. If, given any sequence $F, F_1, F_2, \cdots$ of sets of Φ, the relation $F\cdot\liminf F_i\neq 0$ implies $\liminf F_i\subset F$, then the family Φ is called *upper-semi-continuous*. In this case the sets of the family Φ are disjoint. There is a standard way of introducing a topology into the family Φ which leads to a separable metrizable *hyperspace* Φ^*. The family Φ is 0-dimensional if and only if dim $\Phi^*=0$. In particular, Φ is 0-dimensional whenever it is upper-semi-continuous and countable.

(VI) Let Y be a subset of a topological space X and let Y be homeomorphic with a subset of the linear continuum. The family Φ of the components of Y is 0-dimensional.

The purpose of this note is to establish the following theorem:

THEOREM. *Let X be a unicoherent Peano continuum,*[2] *$\Phi=\{A_\alpha\}$ a 0-dimensional family of subsets of X, and x_1 and x_2 two points of X. If none of the sets A_α cuts X between x_1 and x_2,*[3] *then $\sum_\alpha A_\alpha$ does not cut X between x_1 and x_2.*

Various corollaries can be obtained by taking X to be the n-sphere

[1] Presented to the Society, December 26, 1939, under the title *On 0-dimensional upper-semi-continuous collections.*

[2] A Peano continuum X is called *unicoherent* if given any decomposition $X=X_1+X_2$ into continua, the set $X_1\cdot X_2$ is a continuum.

[3] A set $A\subset X$ *cuts* X between x_1 and x_2 if $X-A$ contains no continuum joining x_1 and x_2.

S^n $(n>1)$ or the n-cube Q^n $(n>0)$ and Φ to be one of the families (I)–(VI).[4]

We shall establish two lemmas before giving the proof of the theorem.

LEMMA 1. *Let X be a unicoherent Peano continuum. Let A_1 and A_2 be open and disjoint subsets of X, and let A_2 be connected. If neither A_1 nor A_2 cuts X between the points x_1 and x_2, then A_1+A_2 does not cut X between x_1 and x_2.*

PROOF. Let C_2 be the component of $X-A_2$ which contains x_1 and x_2. Let $B_1=A_1\cdot C_2$ and $B_2=X-C_2$. It follows easily that B_1 and B_2 are open, $B_1\cdot B_2=0$, B_2 and $X-B_2=C_2$ are connected, and $A_1+A_2\subset B_1+B_2$. It will be sufficient to show that B_1+B_2 does not cut X between x_1 and x_2.

Since $B_1\subset A_1$, the set B_1 does not cut X between x_1 and x_2. We will denote the component of $X-B_1$ which contains x_1 and x_2 by C_1. If $C_1\cdot B_2=0$, then $C_1\subset X-(B_1+B_2)$, so that B_1+B_2 does not cut X between x_1 and x_2. We will suppose then that $C_1\cdot B_2\neq 0$. Since C_1 and B_2 are connected it follows that C_1+B_2 is connected. But $C_1+B_2\subset X-B_1$. Hence $C_1+B_2\subset C_1$, and therefore $B_2\subset C_1$. Since $C_2=X-B_2$, it follows that $X=C_1+C_2$. Since C_1 and C_2 are continua and X is unicoherent, it follows that $C_1\cdot C_2$ is a continuum. But x_1 and x_2 belong to $C_1\cdot C_2$, and $C_1\cdot C_2\subset X-(B_1+B_2)$. Hence B_1+B_2 does not cut X between x_1 and x_2.

LEMMA 2. *Let X be a unicoherent Peano continuum and let $A_1, A_2, \cdots, A_n, \cdots$ be a sequence of disjoint open subsets of X. If none of the sets A_n cuts X between x_1 and x_2, then $\sum_{n=1}^{\infty}A_n$ does not cut X between x_1 and x_2.*

PROOF. Since X is locally connected and separable, every open set in X consists of a countable number of components each of which is open. It is clear, then, that we may assume each set A_n to be connected.

Let k be any positive integer. Using Lemma 1, it follows by finite induction that $A_1+A_2+\cdots+A_k$ does not cut X between x_1 and x_2. Let C_k be the component of $X-(A_1+A_2+\cdots+A_k)$ which contains x_1 and x_2. Then $\prod_{k=1}^{\infty}C_k$ is a continuum containing x_1 and x_2, and $\prod_{k=1}^{\infty}C_k\subset X-\sum_{k=1}^{\infty}A_k$. Hence $\sum_{k=1}^{\infty}A_k$ does not cut X between x_1 and x_2.

[4] See R. L. Moore, Proceedings of the National Academy of Sciences, vol. 10 (1934), p. 356, and S. Eilenberg, Fundamenta Mathematicae, vol. 26 (1936), pp. 76–77.

We now return to the *proof of the theorem*:

For any F in Φ there is an open set $U(F)$ such that $F \subset U(F)$, and $U(F)$ does not cut X between x_1 and x_2. Since Φ is 0-dimensional, there is an open set $V(F)$ such that $F \subset V(F) \subset U(F)$ and $[\overline{V(F)} - V(F)] \sum F = 0$. By the Lindelöf covering theorem, there is a sequence $F_1, F_2, \cdots$ of elements of Φ such that $\sum F \subset \sum_{i=1}^{\infty} V(F_i)$. Now let

$$A_1 = V(F_1), \qquad A_2 = V(F_2) - \overline{V(F_1)},$$
$$A_k = V(F_k) - \overline{[V(F_1) + \cdots + V(F_{k-1})]}, \cdots .$$

The sets $A_1, A_2, \cdots, A_k, \cdots$ are open and disjoint, and no one of them cuts X between x_1 and x_2. But, as is easily shown, $\sum F \subset \sum_{k=1}^{\infty} A_k$. Hence in view of Lemma 2, $\sum F$ does not cut X between x_1 and x_2.

THE UNIVERSITY OF MICHIGAN

SUMS OF FOURTH POWERS OF GAUSSIAN INTEGERS

IVAN NIVEN

It is the purpose of this note to give necessary and sufficient conditions for the expressibility of a Gaussian integer as a sum of fourth powers of Gaussian integers; and then to determine an upper bound to the number of fourth powers necessary when the conditions are satisfied. Our results are as follows:

THEOREM. *A Gaussian integer is expressible as a sum of fourth powers of Gaussian integers if and only if its imaginary coordinate is divisible by* 24. *Every integer* $a+24bi$, *where* a *and* b *are rational integers, is expressible as a sum of* 18 *or fewer fourth powers.*

First we prove that the condition is necessary. We note that[1]

$$(x + yi)^4 = x^4 - 6x^2y^2 + y^4 + 4ixy(x^2 - y^2). \tag{1}$$

It is obvious that $xy(x^2-y^2)$ is divisible by 2 and by 3. Hence any fourth power has an imaginary coordinate divisible by 24, and any sum of fourth powers has the same property.

The converse of this is included in the second statement in the theorem, which we now proceed to prove. The author[2] has shown that a Gaussian integer $a+2bi$ is expressible as a sum of two squares

[1] Latin letters will represent rational integers throughout this paper.

[2] *Integers of quadratic fields as sums of squares*, Transactions of this Society, vol. 48 (1940), p. 410, Theorem 2.

if and only if not both $a/2$ and b are odd rational integers. We use this result in the following two lemmas.

LEMMA 1. *The Gaussian integer $6(c+di)^2$, where c and d are odd and even, respectively, is expressible as a sum of six fourth powers.*

By the theorem stated above, we can write $c+di=\alpha^2+\beta^2$, where α and β are Gaussian integers. Also we employ the identity

$$6(\alpha^2+\beta^2)^2 = 2(\alpha+\beta)^4 + 2(\alpha-\beta)^4 + (\alpha+\beta i)^4 + (\alpha-\beta i)^4,$$

which completes the proof.

LEMMA 2. *A Gaussian integer of the form $48h+6k+24mi$, where k equals 2 or 6 according as m is even or odd, is expressible as a sum of twelve fourth powers.*

As above, we can write

$$8h + k + 4mi = (c + di)^2 + (e + fi)^2, \tag{2}$$

which implies

$$8h + k = c^2 + e^2 - d^2 - f^2, \qquad 2m = cd + ef. \tag{3}$$

Since $8h+k\equiv 2 \pmod 4$, the first of equations (3) shows that either c and e are odd and d and f are even, or vice versa. We show that the latter cannot be the case.

First let m be even, so that $k=2$. Then equations (3) imply the congruences

$$c^2 + e^2 - d^2 - f^2 \equiv 2 \pmod 8, \qquad cd + ef \equiv 0 \pmod 4. \tag{4}$$

Since the square of an odd integer is congruent to 1 modulo 8, the assumption that d and f are odd leads to the congruence $c^2+e^2\equiv 4 \pmod 8$. Hence the integers c and e are even, but are incongruent modulo 4. But these conditions on c and e are incompatible with the second of the congruences (4).

On the other hand, if m is odd, $k=6$, and corresponding to (4) we have the congruences

$$c^2 + e^2 - d^2 - f^2 \equiv 6 \pmod 8, \qquad cd + ef \equiv 2 \pmod 4. \tag{5}$$

Assuming again that d and f are odd, we obtain $c^2+e^2\equiv 0 \pmod 8$. Hence c and e are even, and, moreover, are congruent modulo 4. These conditions are such that the second of congruences (5) has no solutions.

Thus we have proven that d and f are even, and c and e are odd.

Multiplying equation (2) by 6, we see that Lemma 1 is applicable to each term on the right side, and we have the desired result.

To complete the proof of the theorem, we prove this lemma:

LEMMA 3. *Any integer $a+24bi$ is expressible as a sum of six fourth powers and an integer of the type described in Lemma 2.*

We identify b with m, and show that a is congruent to six real fourth powers modulo $48h+6k$. This we do by exhibiting the numbers $0, 1, \cdots, 47$ as sums of fourth powers modulo 48. Most of these can be handled by the use of: $1^4=1$, $2^4=16$, $(1+i)^4=-4$. Except for the values 7, 22, 23, 27, 37, 38, 39, 42, 43, and 47, all integers from 1 to 47 can be expressed as sums of 1, 16, and -4, not more than six summands being used in each case. For example, we have

$$11 = 16 - 4 - 4 + 1 + 1 + 1, \quad 46 = 16 + 16 + 16 - 4 + 1 + 1.$$

Turning now to the exceptional cases, we make use of $81=3^4$, and introduce congruences modulo 48. We can write

$$\begin{aligned} 7 &\equiv 151 = 81 + 81 - 4 - 4 - 4 + 1, \\ 22 &\equiv 70 = 81 - 4 - 4 - 4 + 1, \\ 27 &\equiv 75 = 81 - 4 - 4 + 1 + 1, \\ 37 &\equiv -11 = -4 - 4 - 4 + 1, \\ 42 &\equiv -6 = -4 - 4 + 1 + 1, \\ 47 &\equiv -1 = -4 + 1 + 1 + 1, \end{aligned}$$

these being congruences modulo 48. Since the integer 37 is represented here as a sum of four fourth powers modulo 48, the integers 38 and 39 are similarly sums of five and six fourth powers, respectively. Also the integers 23 and 43 can be compared with 22 and 42 above.

Although the theorem has been proved completely, we now show that Lemma 3 cannot be improved, that is, that six fourth powers are necessary in at least one case. Consider the situation when $a=19+48A$, $b=2B$. In this case we can show that it is not possible to obtain rational integers h and m, and Gaussian integers $\alpha_1, \cdots, \alpha_5$ to satisfy

$$a + 24bi = \sum_{i=1}^{5} \alpha_i^4 + 48h + 6k + 24mi, \tag{6}$$

where, of course, k is 2 or 6 according as m is even or odd.

LEMMA 4. *If the imaginary part of the fourth power of a Gaussian integer is congruent to* 0 *modulo* 48, *the real part is congruent to* 0, 1, *or* 12

modulo 16. *If the imaginary part is congruent to* 24 *modulo* 48, *the real part is congruent to* 9 *modulo* 16.

From equation (1) it follows that

$$R(x+yi)^4 = (x^2-y^2)^2 - 4x^2y^2, \quad I(x+yi)^4 = 4xy(x^2-y^2).$$

Since the square of a rational integer is congruent to 0, 1, 4 or 9 modulo 16, it is easily verified that $R(x+yi)^4$ is congruent to 0, 1, 9, or 12 modulo 16. Also note that $R(x+yi)^4$ is congruent to 9 modulo 16 if and only if x is odd and y is congruent to 2 modulo 4, or vice versa; and when x and y satisfy these conditions, $I(x+yi)^4$ is congruent to 24 modulo 48. In all other cases either xy or x^2-y^2 is divisible by 4, and hence $I(x+yi)^4$ is congruent to 0 modulo 48.

We now use Lemma 4 to prove equation (6) impossible. First let m be even, so that $k=2$. Since b is even the imaginary parts of (6) can be equal only if $\sum\alpha_i^4$ has an imaginary coordinate congruent to 0 modulo 48. This implies that an even number of the terms $I(\alpha_i^4)$ are congruent to 24 modulo 48. In turn, Lemma 4 states that an even number of the terms $R(\alpha_i^4)$ are congruent to 9 modulo 16. The real parts of equation (6) can be written as a congruence:

$$19 \equiv \sum_{i=1}^{5} R(\alpha_i^4) + 12 \pmod{16}.$$

Simple verification shows that this congruence cannot be satisfied by assigning the values 0, 1, 9, or 12 to the terms $R(\alpha_i^4)$, the value 9 being used an even number of times.

Second let m be odd, so that $k=6$. Corresponding to the above congruence we have

$$19 \equiv \sum_{i=1}^{5} R(\alpha_i^4) + 36 \pmod{16}.$$

In this case, however, Lemma 4 requires that we have an odd number of the terms $R(\alpha_i^4)$ congruent to 9 modulo 16. Again it can be verified that the congruence has no solutions satisfying this condition.

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THE GEOMETRY OF WHIRLS AND WHIRL-MOTIONS IN SPACE[1]

J. M. FELD

1. **Introduction.** The geometry of *whirls* and *whirl-motions* in the plane was inaugurated by E. Kasner.[2] An adaptation of Kasner's geometry to the sphere was made by K. Strubecker.[3] It is the purpose of this paper to develop a strictly analogous geometry in euclidean three-space, S_3. To render such a development possible we shall introduce a new type of *oriented* plane element—namely, a geometric object formed by a plane, a point in the plane, and an ordered pair of orthogonal fundamental directions in the plane; the fundamental directions shall be given by a pair of unit vectors in the plane. There are $2\infty^6$ such plane elements in S_3; there are ∞^5 of the Lie kind. Henceforth, *plane element* shall mean only the new kind of plane element.[4]

2. **Turns, slides, and direct whirls.** Let e_0, e_1, e_2, e_3 be the Hamiltonian quaternion units such that

$$e_0e_i = e_ie_0 = e_i, \qquad e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1.$$

Let any real point P in S_3 whose orthogonal cartesian coordinates are z_1, z_2, z_3 be represented by the position vector $z = z_1e_1 + z_2e_2 + z_3e_3$. Let

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[1] Presented to the Society February, 22, 1941.

[2] Edward Kasner, *The group of turns and slides and the geometry of turbines*, American Journal of Mathematics, vol. 33 (1911), pp. 193–202. Further development of the subject appears in a series of papers by Kasner and De Cicco; see, for example, their joint papers, *Quadratic fields in the geometry of the whirl-motion group* G_6, ibid., vol. 61 (1939), pp. 131–142; and *The geometry of the whirl-motion group* G_6: *elementary invariants*, this Bulletin, vol. 44 (1938), pp. 399–403.

[3] K. Strubecker, *Zur Geometrie sphärischer Kurvenscharen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 44 (1934), pp. 184–198.

[4] Inasmuch as our *oriented plane element* defines a position of a rigid body in space, it is essentially equivalent to Study's *soma*, *Geometrie der Dynamen*, Leipzig, 1903, and to De Saussure's *feuillet*, *Exposé Résumé de la Géométrie des Feuillets*, Geneva, 1910; see also R. Bricard, Nouvelles Annales de Mathématiques, (4), vol. 10 (1910). It was remarked by Kasner in his 1911 paper, loc. cit., that it was possible to obtain in any space of constant curvature a group analogous to his group of whirls in the plane and that, moreover, for ordinary space the *feuillet*, consisting of a point, line, and plane all incident with one another, would be an appropriate element. Another generalization of Kasner's turbine geometry, along lines different from those pursued in this paper, has been carried out by A. Narasinga Rao, *Studies in turbine geometry* I, Journal of the Indian Mathematical Society, vol. 3 (1938), pp. 96–108; II, Proceedings of the Indian Academy of Sciences, vol. 8A (1938), pp. 179–186.

OR be an axis through the origin parallel to a given unit vector v, and let

$$\zeta = -\cos\theta/2 + v\sin\theta/2, \qquad \bar{\zeta} = -\cos\theta/2 - v\sin\theta/2.$$

Then the rotation of z around *OR* through the angle θ is given by Hamilton's formula[5]

$$z^* = \bar{\zeta} z \zeta. \tag{2.1}$$

We shall refer to (2.1) as the rotation ζ.

Let $\mathfrak{E}$ be a plane element characterized by the point z lying in the plane P and the pair of fundamental orthogonal directions given by the unit vectors v_1, v_2, in this order. Let ζ be the (unit) quaternion by means of which the vectors e_1 and e_2 can be rotated around an axis passing through the origin into positions parallel respectively to the directions of v_1 and v_2. There evidently exists a (1, 1) correspondence between the plane elements $\mathfrak{E}$ and the pairs of quantities z, ζ; we shall, accordingly, call z, ζ the coordinates of $\mathfrak{E}$, and shall use the symbol (z, ζ) to designate the plane element whose coordinates are z, ζ.

DEFINITIONS. *A slide is a plane element transformation,* $(z, \zeta) \rightarrow (z^*, \zeta^*)$, *which leaves the planes of the plane elements unaltered, but subjects their points to the same translation relative to their fundamental directions.*

A turn is a plane element transformation which leaves the points of the plane elements unaltered, but rotates their planes through the same angle around an axis inclined in the same manner relative to their fundamental directions.

A direct whirl[6] *is a plane element transformation resulting from the product of an arbitrary turn followed first by an arbitrary slide and then by a second arbitrary turn.*

These definitions are the space analogues of those given by Kasner for the plane.

Let $a = a_1 e_1 + a_2 e_2$, (a_i real); then the slide $(z, \zeta) \rightarrow (z^*, \zeta^*)$, which shall be represented by S_a, is given by the equations:

$$z^* = z + \bar{\zeta} a \zeta, \qquad \zeta^* = \zeta. \tag{2.2}$$

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[5] It will suffice for our purpose to restrict ourselves throughout this paper to *unit* quaternions—that is, those having a norm equal to unity.

[6] The term *direct whirl* is used here instead of Kasner's *whirl* because it is necessary to distinguish this from other types of whirl that appear below. This term was first used, for a similar reason, in J. M. Feld's *Whirl-similitudes, euclidean kinematics, and non-euclidean geometry.* (This Bulletin, abstract 46-5-270.)

Let $\alpha=\alpha_0\mathfrak{e}_0+\alpha_1\mathfrak{e}_1+\alpha_2\mathfrak{e}_2+\alpha_3\mathfrak{e}_3$ and $N(\alpha)=1$; then the turn T_α is given by the equations:

$$\zeta^* = z, \qquad \zeta^* = \alpha\zeta. \tag{2.3}$$

The direct whirl $T_\alpha S_a T_\beta$ is given by the equations:

$$z^* = \bar{\zeta}\bar{\alpha}a\alpha\zeta, \qquad \zeta^* = \beta\alpha\zeta.$$

If we let $\bar{\alpha}a\alpha=c$ and $\beta\alpha=\gamma$, the equations of a generic direct whirl $W^+_{c,\gamma}$ assume the form:

$$z^* = z + \bar{\zeta}c\zeta, \qquad \zeta^* = \gamma\zeta \tag{2.4}$$

where c is a vector and γ is a unit quaternion.

We can now state the following theorems:

THEOREM 1. *The slides constitute a continuous two-parameter abelian group such that* $S_a \cdot S_b = S_{a+b}$.

THEOREM 2. *The turns constitute a continuous three-parameter group such that* $T_\alpha \cdot T_\beta = T_{\beta\alpha}$.

THEOREM 3. *The direct whirls constitute a continuous six-parameter group. A necessary and sufficient condition that* $W^+_{c,\gamma} = W^+_{a,\alpha} \cdot W^+_{b,\beta}$ *is that* $c=a+\bar{\alpha}b\alpha$ *and* $\gamma=\beta\alpha$.

We shall let $\mathfrak{W}^+_6$ represent the group of direct whirls. Noting that the direct whirls $W^+_{a,\alpha}$ where $\alpha=\mathfrak{e}_0$, $a=k\mathfrak{e}_3$ (k real), form a one-parameter continuous group of contact transformations—namely, the group of dilatations D_k—we obtain the following theorem:

THEOREM 4. *Direct whirls can be uniquely represented in the form* $W^+_{c,\gamma} = D_k \cdot S_b \cdot T_\alpha$ *where* $c=k\mathfrak{e}_3+b$, $\gamma=\alpha$.

From equations (2.4) we obtain, by multiplying each member of the first equation on the left by the corresponding member of the second equation and on the right by the conjugate of the corresponding member of the second equation, $\zeta^* z^* \bar{\zeta}^* = \gamma(\zeta z \bar{\zeta})\bar{\gamma} + \gamma c \bar{\gamma}$. Letting $Z^* = \zeta^* z^* \bar{\zeta}^*$, $Z = \zeta z \bar{\zeta}$, $\alpha = \bar{\gamma}$, $a = \gamma c \bar{\gamma}$, we have

$$Z^* = \bar{\alpha} Z \alpha + a, \tag{2.5}$$

which represents a euclidean point displacement $Z \to Z^*$ in S_3. Let this displacement be represented by $D^+_{a,\alpha}$. Then we obtain this theorem:

THEOREM 5. *The group of direct whirls is simply isomorphic to the group of euclidean displacements in* S_3: *the correspondence between members of these groups is given by* $D^+_{a,\alpha} \leftrightarrow W^+_{b,\beta}$ *where* $b=\alpha a\bar{\alpha}$ *and* $\beta=\bar{\alpha}$.

3. **The group of whirl-motions.** Equation (2.5), which represents a point transformation, can be extended to represent a plane element transformation $(z, \zeta) \rightarrow (z^*, \zeta^*)$ as follows:

$$z^* = \bar{\alpha} z \alpha + a, \qquad \zeta^* = \zeta \alpha. \tag{3.1}$$

Henceforth, $D^+_{a,\alpha}$ shall designate the euclidean displacement of plane elements given by (3.1); $\mathfrak{D}_6^+$ shall designate the six-parameter group of these displacements. The product of a displacement $D^+_{a,\alpha}$ and a direct whirl $W^+_{b,\beta}$, in either order, shall be called a *direct whirl-motion.* The equations of the direct whirl-motion $D^+_{a,\alpha} \cdot W^+_{b,\beta}$ are:

$$z^* = \bar{\alpha}(z + \bar{\zeta} b \zeta)\alpha + a, \qquad \zeta^* = \beta \zeta \alpha. \tag{3.2}$$

Evidently $D^+_{a,\alpha} \cdot W^+_{b,\beta} = W^+_{b,\beta} \cdot D^+_{a,\alpha}$. The direct whirl-motions constitute a continuous twelve-parameter group, which shall be designated by $\mathfrak{G}^1_{12}$.

Let l and r be an arbitrary pair of vectors, and let the plane element $\mathfrak{E}$: $(l+r, e_0)$ be subjected to the ∞^3 possible rotations around the point whose position vector is equal to l; then the manifold of ∞^3 plane elements (z, ζ) into which $\mathfrak{E}$ is thereby transformed shall be called a (space) *turbine.* The equation of this turbine is

$$z - l = \bar{\zeta} r \zeta. \tag{3.3}$$

Since the vectors l and r completely characterize the turbine (3.3), we shall call l and r its left and right coordinates respectively, and let the symbol $[l, r]$ represent the turbine whose equation is (3.3). A brief computation will verify this theorem:

THEOREM 6. *Turbines are transformed into turbines under* $\mathfrak{G}^1_{12}$. *Under* $\mathfrak{W}_6^+$ *their left coordinates remain invariant; under* $\mathfrak{D}_6^+$ *their right coordinates remain invariant.*

Let us now consider a space analogue of the *flat field*, a concept that appears in Kasner's whirl-motion geometry in the plane. We shall call this analogue a *dual element* because it will serve as the dual of the plane element, just as the flat field serves as the dual of the lineal element.[7] Dual elements shall be represented by two coordinates s and σ where the former is a vector and the latter is a unit quaternion; in order to distinguish dual elements from plane elements, the former shall be represented by the symbol $\{s, \sigma\}$. We define dual elements by imposing upon them the requirement that the turbine $[l, r]$ which, in plane element coordinates z, ζ, has the equation (3.3) shall, in dual coordinates s, σ, have the equation

[7] Feld, loc. cit.

$$(3.3^*) \qquad s - l = -\bar{\sigma} r \sigma.$$

A turbine $[l, r]$ and a plane element (z, ζ) shall be said to be *incident* if their coordinates are connected by equation (3.3); similarly, $[l, r]$ and $\{s, \sigma\}$ shall be said to be incident if their coordinates are connected by equation (3.3*).

It is evident that direct whirls and displacements of plane elements induce contragredient transformations of the dual elements; for example, the dual element transformation contragredient to $W^+_{a,\alpha}$ is

$$(3.4) \qquad s^* = s - \bar{\sigma} a \sigma, \qquad \sigma^* = \alpha\sigma.$$

This can be regarded, of course, as a representation of $W^+_{a,\alpha}$ in dual element coordinates; likewise, displacements and whirl-motions can be expressed in dual element coordinates.

Let $\mathfrak{J}_1$ represent the involutory plane element transformation

$$(3.5) \qquad z^* = -z, \qquad \zeta^* = \zeta,$$

which induces the dual element transformation

$$(3.5^*) \qquad s^* = -s, \qquad \sigma^* = \sigma.$$

The ∞^6 transformations $\mathfrak{J}_1\mathfrak{W}_6^+$ or $\mathfrak{W}_6^+\mathfrak{J}_1$ shall be called *opposite whirls*; the family of opposite whirls shall be designated by $\mathfrak{W}_6^-$. Similarly, the family $\mathfrak{J}_1\mathfrak{D}_6^+ \equiv \mathfrak{D}_6^+\mathfrak{J}_1$, which embraces the ∞^6 euclidean symmetries in S_3, shall be designated by $\mathfrak{D}_6^-$; and the twelve-parameter family yielded by the products of $\mathfrak{J}_1$ and the direct whirl-motions shall be called the family of *opposite whirl-motions*, and shall be represented by $\mathfrak{G}_{12}^2$. Evidently, $\mathfrak{D}_6^-\mathfrak{W}_6^+ \equiv \mathfrak{D}_6^+\mathfrak{W}_6^- \equiv \mathfrak{G}_{12}^2$. Since the product of two opposite whirl-motions is a direct whirl-motion, the two families $\mathfrak{G}_{12}^1$ and $\mathfrak{G}_{12}^2$ form a mixed group.

Let $\mathfrak{J}_2$ represent the involutory correlation $(z, \zeta) \leftrightarrow \{s, \sigma\}$ given by the equations:

$$(3.6) \qquad z = s, \qquad \zeta = \sigma.$$

Let the family of transformations embraced by $\mathfrak{J}_2\mathfrak{G}_{12}^1 \equiv \mathfrak{G}_{12}^1\mathfrak{J}_2$ be represented by $\mathfrak{G}_{12}^3$; likewise, let $\mathfrak{J}_2\mathfrak{G}_{12}^2 \equiv \mathfrak{G}_{12}^2\mathfrak{J}_2$ be represented by $\mathfrak{G}_{12}^4$. The four families $\mathfrak{G}_{12}^i$, $(i = 1, 2, 3, 4)$, constitute a mixed group, which shall be designated by Γ_{12} and called the *group of proper whirl-motions*.

Let $\mathfrak{J}_3$ represent the involutory plane element transformation $(z, \zeta) \rightarrow (z^*, \zeta^*)$ given by the equations:

$$(3.7) \qquad z^* = \zeta z \bar{\zeta}, \qquad \zeta^* = \bar{\zeta}.$$

This transformation induces the dual element transformation

$$(3.7^*) \qquad s^* = -\sigma s \bar{\sigma}, \qquad \sigma^* = \bar{\sigma}.$$

Let the products $\mathfrak{I}_i\mathfrak{G}_{12}^i \equiv \mathfrak{G}_{12}^i\mathfrak{I}_i$, $(i=1, 2, 3, 4)$, be designated by $\mathfrak{H}_{12}^i$. The four families $\mathfrak{H}_{12}^i$ embrace the *improper whirl-motions*. The proper and improper whirl-motions constitute a mixed twelve-parameter group composed of eight mutually exclusive families; this group, which shall be designated by Γ_{12}, shall be called the *complete group of whirl-motions*. The following theorems are now evident.

THEOREM 7. *Turbines are transformed into turbines under the complete group of whirl-motions.*

THEOREM 8. *The turbine transformation effected by $\mathfrak{I}_3$ is given by $l^*=r$, $r^*=l$.*

4. **The kinematic representation.** Let the turbine $[l, r]$ be mapped on that ordered pair of points in S_3 whose position vectors are l and r, in this order. By means of this mapping, which shall be known as the *kinematic representation*,[8] a (1, 1) correspondence is set up between turbines $[l, r]$ and ordered point pairs l, r, the former of which is the left image point and the latter the right image point of $[l, r]$. If we regard, as we may, the $8\infty^{12}$ whirl-motions in Γ_{12} as turbine transformations, $[l, r]\rightarrow[l^*, r^*]$, we find that the kinematic representation maps these transformations (1, 1) upon pairs of euclidean motions (displacements and symmetries) in S_3, which are either of the type $l\rightarrow l^*$, $r\rightarrow r^*$, or else of the type $l\rightarrow r^*$, $r\rightarrow l^*$. The pairs of motions of the former type correspond to the proper whirl-motions, and those of the latter type to the improper whirl-motions. Explicitly, the kinematic images of the families $\mathfrak{G}_{12}^i$ are as follows:

$$
\begin{aligned}
&\mathfrak{G}_{12}^1\colon \quad l^* = \bar{\alpha}l\alpha + a, && r^* = \bar{\beta}r\beta + b,\\
&\mathfrak{G}_{12}^2\colon \quad l^* = -\bar{\alpha}l\alpha + a, && r^* = -\bar{\beta}r\beta + b,\\
&\mathfrak{G}_{12}^3\colon \quad l^* = \bar{\alpha}l\alpha + a, && r^* = -\bar{\beta}r\beta + b,\\
&\mathfrak{G}_{12}^4\colon \quad l^* = -\bar{\alpha}l\alpha + a, && r^* = \bar{\beta}r\beta + b,
\end{aligned}
\tag{4.1}
$$

α and β are unit quaternions; a and b are vectors.

To obtain the kinematic image of the family $\mathfrak{H}_{12}^i$ we need but, in ac-

[8] A similar representation of whirl-motion geometry in the plane has been given by Feld, loc. cit. See also, in this connection, J. Grünwald's *Ein Abbildungsprinzip, welches die ebene Geometrie und Kinematik mit der räumlichen Geometrie verknüpft*, Sitzungsberichte der Akademie der Wissenschaften, Vienna, Mathematisch-Naturwissenschaftliche Klasse (II A), vol. 80 (1911), pp. 677–741; and two contributions by W. Blaschke, *Euklidische Kinematik und nichteuklidische Geometrie*, Zeitschrift für Mathematik und Physik, vol. 60 (1911), pp. 61–91, 203–204; and *Ebene Kinematik*, Berlin, 1938.

cordance with Theorem 8, interchange l and r in the pair of equations given by (4.1) which corresponds to $\mathfrak{G}_{12}^{i}$.

Inasmuch as the turbine $[l, r]$ has for its equation in plane element coordinates $z, \zeta: l = -\bar{\zeta} r \zeta + z$, and for its equation in dual element coordinates $s, \sigma: l = \bar{\sigma} r \sigma + s$, it is apparent that a necessary and sufficient condition that $[l, r]$ be incident to a given plane element (z, ζ) [dual element $\{s, \sigma\}$] is that the left and right image points of $[l, r]$ in the kinematic representation be corresponding points in a euclidean symmetry [displacement]. Consequently, the kinematic representation establishes a (1, 1) correspondence between plane elements [dual elements] and euclidean symmetries [displacements] in S_3.

The kinematic mapping of plane elements and dual elements upon symmetries and displacements, respectively, furnishes us with the means of representing parametrically the composition of euclidean motions in S_3.[9] An example will suffice to show how this can be done. Let the coordinates of the dual element $\{s, \sigma\}$ represent the displacement that corresponds to $\{s, \sigma\}$ in the kinematic representation; let the displacement corresponding to $\{s_3, \sigma_3\}$ be equivalent to the product of the displacements corresponding to $\{s_1, \sigma_1\}$ and $\{s_2, \sigma_2\}$, and in this order. With these conventions it follows that

$$s_3 = \bar{\sigma}_2 s_1 \sigma_2 + s_2, \qquad \sigma_3 = \sigma_1 \sigma_2;$$

in a similar manner we can represent other combinations of euclidean motions.

5. **The group of whirl-similitudes.** Let $\mathfrak{S}$ represent the pair of similitudes:

$$l^* = kl, \qquad r^* = kr$$

where k is a positive real number, and l, r are the kinematic image points of $[l, r]$. Let $\mathfrak{G}_{13}^{i} \equiv \mathfrak{S}\mathfrak{G}_{12}^{i}$ and $\mathfrak{H}_{13}^{i} \equiv \mathfrak{S}\mathfrak{H}_{12}^{i}$. Then the eight thirteen-parameter families comprising $\mathfrak{G}_{13}^{i}$ and $\mathfrak{H}_{13}^{i}$, $(i = 1, 2, 3, 4)$, constitute a mixed group Γ_{13}—the *complete group of whirl-similitudes*. This group is the space analogue of the group of whirl-similitudes in the plane.[10] Evidently, turbines are transformed into turbines under Γ_{13}.

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[9] E. Study has given such a representation by means of Clifford biquaternions, op. cit.

[10] Feld, op. cit.

A THEOREM CONCERNING THE GEODESICS ON A PARABOLOID OF REVOLUTION[1]

DONALD LING AND LEON RECHT

We shall examine the behavior in the large of the geodesics on a paraboloid of revolution. It will appear that on every geodesic there is a unique point—the "point of symmetry"—which divides the geodesic into two "conjugate rays," which are mirror images of each other in the plane determined by that point and the axis of the paraboloid. Every geodesic except the meridians, which have no singularities whatever, has infinitely many double points, but no other singular points. If, starting at the point of symmetry, we imagine the conjugate rays of a geodesic to be traversed simultaneously by two points which are images in the plane of symmetry, and which continue until they meet for the first time—at the "first double point" of the geodesic—we shall have generated a "loop of type 1." If, starting at the point of symmetry, they pass through the first double point and continue until they next meet—at the "second double point"—we shall have formed a "loop of type 2." We define in this way loops of types 1, 2, 3, $\cdots$. The last double point to be reached in the process of generating a loop of a given type will be called the "vertex" of the loop.

The main result of this paper is the following: The paraboloid can be divided by planes perpendicular to the axis into infinitely many zones, each zone to include the points of that bounding circle which is nearer the vertex. The zone containing the vertex we call the "cap," the following the "first zone," and so on. The zones have the following properties:

1. Each interior point of the kth zone is the vertex of exactly two distinct loops of each of the types 1, 2, $\cdots$, k, but is the vertex of no loop of type greater than k.

2. Each boundary point in the kth zone is the vertex of two loops of each of the types 1, 2, $\cdots$, $k-1$ and of one loop of type k. It is the vertex of no loop of type greater than k.

3. No point of the cap is the vertex of any loop.

Let the paraboloid be that obtained by revolving $r^2=x$ about the x-axis. As coordinates of a point P on the surface (the vertex excepted) we take r to be the length of the vector R from P perpendicular to the axis and directed toward P, and θ to be the angle which the plane determined by P and the axis makes with an arbitrary plane contain-

[1] Presented to the Society, February 24, 1940.

ing the axis. The differential equation defining the geodesics (cf. Darboux, *Leçons sur la Théorie Générale des Surfaces* · · · , vol. 3) is

$$\frac{d\theta}{dr} = -\frac{c}{r}\left[\frac{4r^2+1}{r^2-c^2}\right]^{1/2} \tag{1}$$

where c is an arbitrary constant. Corresponding to $c=0$ are the meridians $\theta=\text{const}$. Since their nature is obvious we may except them from consideration, and assume $c\neq 0$. The solution of (1) corresponding to a given value of $c\neq 0$ and passing through (r_0, θ_0) is

$$\begin{aligned}\theta = \theta_0 &+ \arctan c\left[\frac{4r^2+1}{r^2-c^2}\right]^{1/2} - c\log\frac{(4r^2+1)^{1/2}+2(r^2-c^2)^{1/2}}{(4r^2+1)^{1/2}-2(r^2-c^2)^{1/2}} \\ &- \arctan c\left[\frac{4r_0^2+1}{r_0^2-c^2}\right]^{1/2} + c\log\frac{(4r_0^2+1)^{1/2}+2(r_0^2-c^2)^{1/2}}{(4r_0^2+1)^{1/2}-2(r_0^2-c^2)^{1/2}} \\ &= \theta_0 + f(r; r_0, c), \text{ say.}\end{aligned} \tag{2}$$

As $r\to|c|$ the right member of (2) approaches

$$\begin{aligned}\bar\theta = \theta_0 &+ \frac{\pi}{2}\operatorname{sign} c - \arctan c\left[\frac{4r_0^2+1}{r_0^2-c^2}\right]^{1/2} \\ &+ c\log\frac{(4r_0^2+1)^{1/2}+2(r_0^2-c^2)^{1/2}}{(4r_0^2+1)^{1/2}-2(r_0^2-c^2)^{1/2}}.\end{aligned} \tag{3}$$

It can be shown without difficulty that $(|c|, \bar\theta)$ is a point of the corresponding geodesic.[2] It is apparent from (2) and (3) that if $c<0$ θ increases monotonically from $\bar\theta$ at $r=|c|$ to ∞ at $r=\infty$, while if $c>0$ θ decreases monotonically from $\bar\theta$ at $r=|c|$ to $-\infty$ at $r=\infty$.

Consider the two solutions

$$\begin{aligned}\theta^{(1)} &= \theta_0 + f(r; r_0, c), \\ \theta^{(2)} &= \{\theta_0 + 2f(|c|; r_0, c)\} + f(r; r_0, -c) \\ &= \{\theta_0 + 2f(|c|; r_0, c)\} - f(r; r_0, c).\end{aligned}$$

It is clear that $\theta^{(1)}(|c|)=\theta^{(2)}(|c|)=\bar\theta$. Moreover, at the point $(|c|, \bar\theta)$ the two solutions have a common tangent, and thus belong to the same geodesic. Furthermore

$$\theta^{(1)}(r) - \bar\theta = f(r; r_0, c) - f(|c|; r_0, c) = -[\theta^{(2)}(r) - \bar\theta] \tag{4}$$

[2] It is readily verified that if r and θ are expressed in terms of the arc length s along the geodesic measured from (r_0, θ_0), then $r\to|c|$ and $\theta\to\bar\theta$ as s approaches a certain finite limit.

so that the points $(r, \theta^{(1)}(r))$ and $(r, \theta^{(2)}(r))$ are mirror images of each other in the plane determined by the axis and the point $(|c|, \bar{\theta})$. This is the point of symmetry of the geodesic, and equations (4) give a pair of conjugate rays as defined above. Bearing in mind that $\theta^{(1)}(r)$ and $\theta^{(2)}(r)$ are monotonic in opposite senses, and that the absolute value of each is large with r, we see that each geodesic has infinitely many double points, but no other multiple points.

Assume $\theta_0 = 0$. Then $\bar{\theta}$ is the angle through which the radius vector R turns as (r, θ) moves along a geodesic ray from its initial point $(r_0, 0)$ to its point of symmetry $(|c|, \bar{\theta})$. We shall study the variation of $\bar{\theta}$ as the parameter c, which controls the direction of the geodesic ray at its initial point $(r_0, 0)$, varies over non-negative values.

We find from (3) for $c > 0$ that

$$\frac{d\bar{\theta}}{dc} = -\lambda + \log\frac{\lambda+2}{\lambda-2}; \qquad \lambda = \left[\frac{4r_0^2+1}{r_0^2-c^2}\right]^{1/2} \tag{5}$$

As λ increases from 2 to ∞ the logarithm in (5) decreases monotonically from ∞ to 0. Thus $d\bar{\theta}/dc = 0$ has a single root $\lambda_0 > 2$. In terms of c, from the second of equations (5), $d\bar{\theta}/dc$ vanishes when and only when $c = c^* = ((\lambda_0^2-4)r_0^2-1)^{1/2}$. This root is real if, as we now assume, $r_0 \geqq 1/(\lambda_0^2-4)^{1/2}$. If $c > c^*$, $d\bar{\theta}/dc < 0$; if $c < c^*$, $d\bar{\theta}/dc > 0$. Since $\bar{\theta}(r_0, c)$ is continuous in c for $0 \leqq c < r_0$, and since $\bar{\theta}(r_0, 0) = \pi/2$ and $\lim_{c \to r_0} \bar{\theta}(r_0, c) = 0$, $\bar{\theta}(r_0, c)$ attains its greatest value, $\theta^*(r_0)$, at $c = c^*$.

From (3) and (5)

$$\begin{aligned}\theta^*(r_0) = \pi/2 &- \arctan \lambda_0((\lambda_0^2-4)r_0^2-1)^{1/2} \\ &+ \lambda_0((\lambda_0^2-4)r_0^2-1)^{1/2}.\end{aligned} \tag{6}$$

This function is continuous for $r_0 \geqq 1/(\lambda_0^2-4)^{1/2}$ and its derivative, $(\lambda_0^2-4)r_0^2-1)^{1/2}/r_0$, is positive for $r_0 \geqq 1/(\lambda_0^2-4)^{1/2}$. Thus $\theta^*(r_0)$, which gives $\bar{\theta}$ for the geodesic ray of greatest $\bar{\theta}$ through $(r_0, 0)$, is continuous and strictly increasing. Moreover $\theta^*(1/(\lambda_0^2-4)^{1/2}) = \pi/2$ and $\lim_{r_0 \to \infty} \theta^*(r_0) = \infty$. There is, then, a sequence of values of r_0: $r_0^{(1)} < r_0^{(2)} < r_0^{(3)} < \cdots$ such that $\theta^*(r_0^{(k)}) = k\pi$ $(k = 1, 2, \cdots)$. The portion of the paraboloid defined by $r_0^{(k)} \leqq r_0 < r_0^{(k+1)}$ we shall call the "kth zone," and the parallel $r_0 = r_0^{(k)}$ its boundary. Let r_0 be an interior point of the kth zone; then $k\pi < \theta^*(r_0) < (k+1)\pi$. As c increases from 0 to c^*, $\bar{\theta}(r_0, c)$ increases from $\pi/2$ to $\theta^*(r_0)$, and as c then increases from c^* to r_0 $\theta(r_0, c)$ decreases from $\theta^*(r_0)$ to 0. Hence there are two sequences of values of c, $\{c_1^{(1)}, \cdots, c_k^{(1)}\}$ and $\{c_1^{(2)}, \cdots, c_k^{(2)}\}$ satisfying $c_1^{(1)} < c_2^{(1)} < \cdots < c_k^{(1)} < c^* < c_k^{(2)} < \cdots < c_1^{(2)}$ and having the property that $\bar{\theta}(r_0, c_m^{(1)}) = \bar{\theta}(r_0, c_m^{(2)}) = m\pi$, $(m = 1, \cdots, k)$. When r_0 is

on the boundary of the kth zone, $c_k^{(1)}=c^*=c_k^{(2)}$; but, for $m<k$, $c_m^{(1)}$ and $c_m^{(2)}$ are distinct.

Interpreted geometrically this means that through a point $(r_0, 0)$ in the kth zone there are, for each value of $m=1, \cdots, k$, two geodesic rays for which $\bar{\theta}=m\pi$. The conjugate ray of each, being a reflection of the original ray, clearly passes through $(r_0, 2m\pi)$, which is identical with $(r_0, 0)$. This point is therefore the vertex of two loops of each of the types $1, \cdots, k$. If $(r_0, 0)$ is on the boundary of the kth zone this statement must be modified to the effect that the loops of type k are coincident, those of lower types remaining distinct.

It remains to examine the cap, that is, that portion of the surface for which $r_0<r_0^{(1)}$. If $1/(\lambda_0^2-4)^{1/2}\leqq r_0<r_0^{(1)}$, $\theta^*(r_0)<\pi$ and no loop with vertex $(r_0, 0)$ is possible. If $r_0<1/(\lambda_0^2-4)^{1/2}$, then $\lambda=[(4r_0^2+1)/(r_0^2-c^2)]^{1/2}\geqq(4+1/r_0^2)^{1/2}>\lambda_0$ and $d\bar{\theta}/dc<0$. As c increases from 0 to r_0, $\bar{\theta}(r_0, c)$ decreases from $\pi/2$ to 0. Thus no loop having a vertex in this portion is possible. We have now established all the results stated in the introduction.

The numbers $r_0^{(n)}$ can be calculated from (6). For, if we set $\theta^*(r_0)=n\pi$ and replace $((\lambda_0^2-4)r_0^2-1)^{1/2}$ by μ, (6) becomes $\mu-\arctan\mu=(2k-1)\pi/2$. Since the left member increases monotonically from 0 to ∞ as μ increases from 0 to ∞, there is exactly one root μ_n for each value of $r=1, 2, 3, \cdots$, and $r_0^{(n)}=[(\mu_n^2+1)/(\lambda_0^2-4)]^{1/2}$. For large values of n, μ_n is of the order of $n\pi$, and $r_0^{(n)}$ is of the order of $[(n^2\pi^2+1)/(\lambda_0^2-4)]^{1/2}$. We have approximately $\lambda_0=2.40$ and $1/(\lambda_0^2-4)^{1/2}=3/4$. Thus $3n\pi/4$ furnishes an estimate of $r_2^{(n)}$ for large values of n.

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PROJECTION OF THE SPACE (m) ON ITS SUBSPACE (c_0)

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In a paper in the Duke Journal, A. E. Taylor[1] remarks that it is an open question whether or not there exists a projection of the space (m), of bounded sequences, on its subspace (c_0), the space of sequences convergent to 0. In this note we make a few remarks which supplement those of Taylor on this question, and we point out that a negative answer follows from a recent result of R. S. Phillips,[2] so that the question is now settled.

Taylor shows that if a projection of the space (c), of convergent sequences, on the space (c_0) exists, it must be of norm greater than or equal to 2. This implies the same result for (m) on (c_0), since any projection of (m) on (c_0) would be in particular a projection of (c) on (c_0).

The space (c) is essentially of dimension only one greater than that of its subspace (c_0). This follows since (c) is obviously the set of all elements of the form $x=x^{(0)}+tX_1$, where $X_1=(1, 1, \cdots)$, $x^{(0)}\in(c_0)$, and t is a number. If $x=\{x_i\}$ is any element of (c), the linear functional $a(x)=t=\lim_{n\to\infty} x_n$ is of norm 1, and vanishes on the subspace (c_0). Now it is a remark of Bohnenblust[3] that for any subspace of a normed linear space L defined by the vanishing of a fixed linear functional on L, there exist projections of norm less than or equal to $2+\epsilon$, for arbitrary $\epsilon>0$. Consequently there are projections of (c) on (c_0) of norm less than or equal to $2+\epsilon$.

There are projections of (c) on (c_0) which are of norm exactly 2, as may be seen as follows. If $x=(x^{(0)}+tX_1)\in(c)$, the general form of a projection of (c) on (c_0) is

$$Px = x + t\{b_i\} = x^{(0)} + t(X_1 + \{b_i\})$$

where $\{b_i\}$ is any sequence of constants such that $\lim_{i\to\infty} b_i=-1$.[4] To calculate the norm of P, we have $\|Px\|\leqq\|x\|+|t|\cdot\sup_i |b_i|$, and $\|x\|=\|x^{(0)}+tX_1\|=\sup_i |x_i^{(0)}+t|\geqq|t|$ since $x_i^{(0)}\to 0$. Therefore $|P|\leqq 1+\sup_i |b_i|$, and because of Taylor's result this has the value

[1] *The extension of linear functionals*, Duke Mathematical Journal, vol. 5 (1939), pp. 538–547; p. 547.

[2] *On linear transformations*, Transactions of this Society, vol. 48 (1940), pp. 516–541; pp. 539–540.

[3] *Convex regions and projections in Minkowski spaces*, Annals of Mathematics, (2), vol. 39 (1938), pp. 301–308; p. 308.

[4] See Taylor, op. cit.

2 for all projections such that $|b_i| \leqq 1$ for all i. In particular the projection $Px = x - a(x) \cdot X_1 = x^{(0)}$, obtained by taking $b_i = -1$ for all i, is of norm 2.

Corresponding to any sequence $\{e_i\}$, where, for each i, e_i is either $+1$ or -1, let us define a space $c_{\pm} \ldots$ as the set of all elements of the form $x + t\{e_i\}$, where $x \in (c_0)$. Any space $c_{\pm} \ldots$ is equivalent[5] to (c) under the automorphism $\{x_i\} \leftrightarrow \{e_i x_i\}$ of (m). Also (c_0) is a subspace of $c_{\pm} \ldots$, since (c_0) is invariant under the automorphism. There are obviously projections of norm 2 of any $c_{\pm} \ldots$ on (c_0), similar to the projections of (c) on (c_0).

If L and L' are any two linear subspaces of (m), we define the *sum* $L + L'$ as the linear subspace of (m) which consists of all elements of the form $x + x'$, where $x \in L$, $x' \in L'$. As usual we denote the closure in (m) of any subspace by a horizontal line over the symbol representing the subspace.

For any set of n linearly independent sequences $X_j = \{e_{ij}\}$, $e_{ij} = +1$ or -1, $j = 1, 2, \cdots, n$, let l_n denote the n dimensional space of all linear combinations $\sum_j t_j X_j$. Suppose the X_j's are such that l_n and (c_0) do not intersect, except in the origin. Then we may define a subspace (c_n) of (m) as the space $(c_0) + l_n$. Similarly, for any countable sequence of X_j's, any finite number of which are linearly independent, we denote the space of all finite linear combinations $\sum_j t_j X_j$ by l_f, and we define $l_\infty = \bar{l}_f$ $(\infty = \aleph_0)$. If l_f and (c_0) intersect only in the origin, we define

$$(c_\infty) = \overline{(c_0) + l_\infty} = \overline{(c_0) + l_f}\,.$$

(If l_∞ and (c_0) intersect only in the origin, each $x \in (c_0) + l_\infty$ has a unique decomposition $x = x^{(0)} + x^{(\infty)}$, $x^{(0)} \in (c_0)$, $x^{(\infty)} \in l_\infty$; otherwise not. The space $(c_0) + l_\infty$ is not necessarily closed.)

Let $X_1 = \{e_{i1}\}$, $X_2 = \{e_{i2}\} \cdots$ be the rows of an infinite matrix $\{\{e_{ij}\}\}$; let $\{\{e_{ij}\}\}_n$ denote the matrix of the first n rows of $\{\{e_{ij}\}\}$. Then any space (c_n) or (c_∞) is determined by a matrix $\{\{e_{ij}\}\}_n$ or $\{\{e_{ij}\}\}$.

THEOREM 1. *In any space* (c_n) [*or* (c_∞)] *such that for every column* C_k *which appears in the matrix* $\{\{e_{ij}\}\}_n$ [*for each* n] *one of* $+C_i$, $-C_i$ *is identical with* C_k *for an infinite number of values of* i, *as* $i \to \infty$, *there exists a projection of* (c_n) *or* (c_∞) *on* (c_0), *which is of norm* 2.

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[5] Two normed linear spaces are isomorphic if there exists a 1-1 transformation T between them which is linear (that is, distributive and continuous) in both directions; they are equivalent if in addition $|T| = |T^{-1}| = 1$.

PROOF. If $x=x^{(0)}+\sum_{j=1}^{j=n} t_j X_j$, $x^{(0)}\in(c_0)$, define $Px=x^{(0)}$. Then

$$\|Px\| = \left\| x - \sum_j t_j X_j \right\| \leqq \|x\| + \left\| \sum_j t_j X_j \right\| = \|x\| + \sup_i \left| \sum_j t_j e_{ij} \right|.$$

By the hypothesis on the matrix $\{\{e_{ij}\}\}_n$, and since if $x^{(0)}=\{x_i^{(0)}\}$, $x_i^{(0)}\to 0$, $\|x\|=\sup_i |x_i^{(0)}+\sum_j t_j e_{ij}| \geqq \max_i |\sum_j t_j e_{ij}|$. This implies $\|Px\| \leqq 2\|x\|$ for any x of (c_n) or of the dense linear set in (c_∞). In the latter case P has a unique continuous extension to (c_∞), with range (c_0), and the norm is preserved. This verifies the theorem.

THEOREM 2. *For every space (c_n), $1\leqq n\leqq\infty$, there is a projection of (c_n) on (c_0) which is of norm* 2.

PROOF. Let (c_n) be determined by $X_1=\{e_{i1}\}$, $X_2=\{e_{i2}\}$, $\cdots$, as above. Consider first the case of any finite n. Since only a finite number of different n-element columns are possible, there can be only a finite number of columns C_k in the matrix $\{\{e_{ij}\}\}_n$ which are not infinitely repeated. Since X_1 and X_2 are linearly independent, and l_2 intersects (c_0) only in the origin, the matrix $\{\{e_{ij}\}\}_2$ evidently satisfies the hypothesis of Theorem 1. Consider $(c_3)=(c_0)+l_3$. By altering at most a finite number of coordinates of X_3, the matrix $\{\{e_{ij}\}\}_3$ can be made to satisfy the hypothesis. Let the altered sequence X_3 be X_3'; then $X_3'=X_3+y^{(0)}$, where $y^{(0)}\in(c_0)$. Therefore the space $(c_3)'$ determined by X_1, X_2, X_3' coincides with (c_3). Proceeding in similar fashion, we see that any space (c_n) or $(c_0)+l_f$ may be represented by a matrix $\{\{e_{ij}\}\}_n$ or $\{\{e_{ij}\}\}$ which *does* satisfy the hypothesis of Theorem 1. Our theorem is thus proved.

THEOREM 3. *There exists a matrix $E=\{\{e_{ij}\}\}$, such that*: (1) *the corresponding subspace l_∞ of (m) intersects (c_0) only in the origin, and is equivalent to the space $l_{1,\infty}$ of absolutely convergent series*; (2) *$(c_0)+l_\infty$ is closed in (m), and l_∞ and (c_0) are complementary subspaces in $(c_\infty)=(c_0)+l_\infty$*; (3) *the projection of (c_∞) on (c_0) according to the complementary subspace l_∞ of (c_0) in (c_∞) is of norm* 2.

PROOF. Let $+$ and $-$ represent $+1$ and -1. The matrix E is

$$\begin{array}{cccccccc}
+ & - & + & - & + & - & + & -\cdots \\
+ & + & - & - & + & + & - & -\cdots \\
+ & + & + & + & - & - & - & -\cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

(The kth row of E consists of alternately 2^{k-1} plus signs and 2^{k-1} minus signs, for all k.) In this matrix, obviously *all* possible finite

combinations of $+$'s and $-$'s occur in the columns. Thus we have for any n $\|\sum_1^n t_j X_j\| = \sum_1^n |t_j|$. The space l_∞ determined by E is therefore equivalent to $l_{1,\infty}$. Suppose that (c_0) and (l_∞) have a common element x. Then there is a sequence $\{x_n\}$, $x_n = \sum_1^n c_{j,n} X_j$, which converges to x. Since $x \in (c_0)$, by the construction of E we must have $\sum_1^n |c_{j,n}| \to 0$ as $n \to \infty$. But this implies $x_n \to 0$, $x = 0$. This verifies statement (1) of Theorem 3; statement (3) follows by Theorem 1. Statement (2) then follows by a lemma of F. J. Murray, that (for Banach spaces) the existence of projections and of complementary manifolds are equivalent.[6]

THEOREM 4. *There exists a subspace of (m) equivalent to the space (C) of continuous functions, such that:* (1) *the subspace (C) of (m) and (c_0) intersect only in the origin*; (2) *$(C)+(c_0)$ is closed in (m)*; (3) *the projection of $(C)+(c_0)$ on (c_0) according to the complementary subspace (C) of (c_0) in $(C)+(c_0)$ is of norm 2.*

PROOF. Let $\{r_i\}$ be the set of all rational numbers of the interval $(0, 1)$, in their usual enumeration. Then any $x = x(t)$ of (C) is uniquely determined by the set of values $\{x(r_i)\}$, while (m) may be regarded as the set of all bounded real-valued functions $x = \{x_i\} = \{x(r_i)\}$ on the set $\{r_i\}$. Clearly this correspondence of each $x(t) \in (C)$ to its set of values $\{x(r_i)\}$ in (m) is an equivalence. Furthermore it is obvious that if $\lim_{i \to \infty} x(r_i) = 0$, $x(t) \equiv 0$. Therefore (C) according to this imbedding intersects (c_0) only in 0.

If $x = \{x(r_i)\}$ corresponds to $x(t) \in (C)$, and $x^{(0)} = \{x^{(0)}(r_i)\}$ belongs to the subspace (c_0) of (m), then $\sup_i |x(r_i) + x^{(0)}(r_i)| \geqq \max |x(t)| = \sup_i |x(r_i)|$ since $x^{(0)}(r_i) \to 0$, and $x(t)$ is continuous. Thus $\|x + x^{(0)}\| \geqq \|x\|$, and the operation defined by $Q(x + x^{(0)}) = x$ is a projection of norm 1 of $(C)+(c_0)$ on (C). The operation $P = I - Q$ is a projection of $(C)+(c_0)$ on (c_0), and $|P| = |I - Q| \leqq |I| + |Q| = 2$. The proof of statement (2) of Theorem 4 is the same as the first part of the proof of Theorem 6, below.

It is a well known theorm of Banach that every separable Banach space is equivalent to a subspace of (C).[7] Therefore by Theorem 4, if (B) is any separable Banach space, there is a subspace of (m) equivalent to (B), such that the projection of $(c_0)+(B)$ on (c_0) through (B) is of norm 2.

[6] See F. J. Murray, Transactions of this Society, vol. 41 (1937), pp. 138–139.

In a normed linear space L, two linear subspaces L_1 and L_2 are called *complementary* manifolds or subspaces if (1) they are closed in L, and intersect only in 0; (2) every $x \in L$ has a decomposition $x = x_1 + x_2$, where $x_1 \in L_1$, $x_2 \in L_2$. It follows by assumption that this decomposition is unique.

[7] S. Banach, *Opérations Linéaires*, p. 185, Theorem 9.

The result of Theorem 2 may be generalized, by removing the restriction that the e_{ij}'s may have only the values $+1$ or -1. Let $X_1=\{e_{i1}\}$ be any element of $(m)-(c_0)$. Let $S_1=[tX_1]$ be the space of all elements tX_1, and let $B_1=(c_0)+S_1$. If P is a projection of B_1 on (c_0), define a sequence of functionals $\{y_i(x)\}$ by $Px=x^{(0)}+tX_1+\{y_i(x)\}$, where $x=x^{(0)}+tX_1$, $x^{(0)}\in(c_0)$. The functionals $y_i(x)$ are obviously linear, and $y_i(x^{(0)})=0$ for all i and $x^{(0)}\in(c_0)$. Let $y_i(X_1)=a_{i1}$. Then $tPX_1=tX_1+t\{a_{i1}\}\in(c_0)$, and we see that the general form of a projection of B_1 on (c_0) is $Px=x^{(0)}+t\{a_{i1}+e_{i1}\}$, where $\{a_{i1}\}$ is any sequence of constants such that $\lim_{i\to\infty}(a_{i1}+e_{i1})=0$.

If $X_1=\{e_{i1}\}$ as above, let $\limsup_i|e_{i1}|=a$. Define $X_1'=\{e_{i1}'\}$ by $e_{i1}'=e_{i1}$ if $|e_{i1}|\leqq a$, $e_{i1}'=a$ sign e_{i1} if $|e_{i1}|\geqq a$. Then $(X_1-X_1')=\{e_{i1}-e_{i1}'\}\in(c_0)$, and the space $B_1'=(c_0)+S_1'$ is the same as B_1. Thus without loss of generality we may always assume $|e_{k1}|\leqq a=\limsup_i|e_{i1}|$ for all k. This assumption is made in the following paragraph.

There are projections of any space B_1 on (c_0) which are of norm 2. For if $Px=x+t\{a_{i1}\}$, then $\|Px\|\leqq\|x\|+|t|\cdot\sup_i|a_{i1}|$. Since $\|x\|=\|x^{(0)}+t\{e_{i1}\}\|=\sup_i|x_i^{(0)}+te_{i1}|\geqq\sup_i|te_{i1}|$, we see that $|P|$ is 2 for all P such that $|a_{i1}|\leqq|e_{i1}|$ for all i. (Note that in no case can $|P|$ be *less* than 2.)

For any finite or countable set of linearly independent sequences $X_j=\{e_{ij}\}$, (e_{ij}'s not restricted to ±1), let S_n or S_f respectively denote the subspace of all linear (finite) combinations $\sum_j t_jX_j$. As before, suppose that S_n or S_f does not intersect (c_0), except in the origin. Define $B_n=(c_0)+S_n$, $B_f=(c_0)+S_f$, $B_\infty=\overline{B_f}$. The general form of a projection of B_n on (c_0) is $Px=x+\{\sum_j t_ja_{ij}\}$ where $x=x^{(0)}+\sum_j t_jX_j$, $x^{(0)}\in(c_0)$, and the a_{ij}'s are any constants such that $\lim_{i\to\infty}(a_{ij}+e_{ij})=0$ for each j. This follows by additivity of P, since P must be in particular a projection of each of the subspaces $(c_0)+[tX_j]$ of B_n on (c_0).

THEOREM 5. *If W is any separable Banach subspace of (m) such that $W\supset(c_0)$, there is a projection of W on (c_0) which is of norm 2.*

PROOF. It follows easily from the hypothesis that W is separable, that there exists a sequence $\{X_j\}$ such that either W is a space B_n (n finite), or

$$W=B_\infty=\overline{(c_0)+S_f}.$$

(Conversely, any space B_n or B_∞ is separable.)

Consider any space B_n or B_f determined by a matrix $E=\{\{e_{ij}\}\}$ which is such that the e_{ij}'s have only a finite number N of different values. By an argument similar to the proofs of Theorems 1 and 2,

we see that there is a projection of B_n or B_f on (c_0) which is of norm 2. This projection is specifically the one given by the general form above when $a_{ij} = -e_{ij}$ for all i, j (after the e_{ij}'s have been suitably altered).

For any space B_n or B_f, without loss of generality we may assume that $\|X_j\| = 1$ for all j. For each integer N, divide the interval $0 \leqq u \leqq 1$ into N subintervals $0 \leqq u \leqq N^{-1}$, $N^{-1} < u \leqq 2N^{-1}$, $\cdots$, $(k-1)N^{-1} < u \leqq kN^{-1}$, $\cdots$, $(N-1)N^{-1} < u \leqq 1$. If $|e_{ij}|$ lies in the kth of these intervals, let $k_{ij} = kN^{-1}$ sign e_{ij}. In this way we associate with the matrix $\{\{e_{ij}\}\}_n$ of B_n a matrix $\{\{k_{ij}\}\}_n$, in which the k_{ij}'s have only the values $\pm kN^{-1}$, $k = 1, \cdots, N$. By a finite number of alterations, the matrix $\{\{k_{ij}\}\}_n$ may be changed into a matrix $\{\{k_{ij}^{(N)}\}\}_n$ in which every column is infinitely repeated; this requires addition of integral multiples of $\pm N^{-1}$ to a finite number of the k_{ij}'s. Alter the matrix $\{\{e_{ij}\}\}_n$ by adding the same multiples of $\pm N^{-1}$ to the corresponding e_{ij}'s; we thus obtain a new matrix $\{\{e_{ij}^{(N)}\}\}_n$ which represents the same space B_n, and which has $\{\{k_{ij}^{(N)}\}\}_n$ for its associated matrix. Let the rows of $\{\{e_{ij}^{(N)}\}\}$ be denoted by $\{X_j^{(N)}\}$; those of $\{\{k_{ij}^{(N)}\}\}$ by $\{Y_j^{(N)}\}$.

If $x \in B_n$, for each N let $x = x^{(0,N)} + \sum_{j=1}^{n} t_j X_j^{(N)}$, where $x^{(0,N)} \in (c_0)$. Let $x^{(0,N)} = P^{(N)}x$. Let N take on only the succession of values 2^ν, $\nu = 1, 2, 3, \cdots$. Then evidently we may make the required alterations in the matrices $\{\{k_{ij}^{(N)}\}\}_n$ (and $\{\{e_{ij}^{(N)}\}\}_n$) in such a way that $\lim_{N,N'} \|X_j^{(N)} - X_j^{(N')}\| = 0$ for each j; that is, $\{X_j^{(N)}\}$ is a Cauchy sequence for each j. If $x = x^{(0,N)} + \sum_j t_j X_j^{(N)}$ and $x = x^{(0,N')} + \sum_j t_j X_j^{(N')}$,[8] then $0 = x^{(0,N)} - x^{(0,N')} + \sum_j t_j (X_j^{(N)} - X_j^{(N')})$, and

$$\|x^{(0,N)} - x^{(0,N')}\| = \Big\|\sum_j t_j (X_j^{(N)} - X_j^{(N')})\Big\|.$$

Therefore for each $x \in B_n$, $\{x^{(0,N)}\}$ is a Cauchy sequence of elements of (c_0).

Let $x^{(0)} = \lim_N x^{(0,N)}$; and define $Px = x^{(0)}$. Then

$$\|P^{(N)}x\| = \Big\|x - \sum_j t_j X_j^{(N)}\Big\| = \Big\|x - \sum_j t_j Y_j^{(N)} + \sum_j t_j (Y_j^{(N)} - X_j^{(N)})\Big\|$$

$$\leqq \|x\| + \Big\|\sum_j t_j Y_j^{(N)}\Big\| + \Big\|\sum_j t_j (Y_j^{(N)} - X_j^{(N)})\Big\|.$$

We also have

[8] The t_j's do not change with N, because in the expression of any x, for each succeeding N the finite alterations can be compensated by changing $x^{(0,N)}$—and for each system $\{X_j^{(N)}\}$, the expression of x is unique (by the hypothesis that S_n intersects (c_0) only in the origin).

$$\|x\| = \left\| x^{(0,N)} + \sum_j t_j X_j^{(N)} \right\|$$

$$\geqq \left\| x^{(0,N)} + \sum_j t_j Y_j^{(N)} \right\| - \left\| \sum_j t_j (Y_j^{(N)} - X_j^{(N)}) \right\|$$

$$\geqq \left\| \sum_j t_j Y_j^{(N)} \right\| - \left\| \sum_j t_j (Y_j^{(N)} - X_j^{(N)}) \right\|$$

by the triangle property of the norm, and since each column of $\{\{k_{ij}^{(N)}\}\}_n$ is infinitely repeated. Therefore $\|P^{(N)}x\| = \|x^{(0,N)}\| \leqq 2\|x\| + 2\|\sum_j t_j(Y_j^{(N)} - X_j^{(N)})\|$. Taking the limit as $N \to \infty$ $(\nu \to \infty)$ in this inequality, we see that $\|Px\| \leqq 2\|x\|$ (since $\|P^{(N)}x - Px\| \geqq |\|P^{(N)}x\| - \|Px\||$ $\to 0$, $\|P^{(N)}x\| \to \|Px\|$).

The operation P, defined for each $x \in B_n$ or B_f, is clearly a projection of norm 2 of B_n or B_f on (c_0), and our theorem is proved. (In case of B_f, as in Theorem 1 P has a unique extension to $\overline{B}_f = B_\infty = W$.)

The foregoing discussion has produced instances of the existence of projections defined on subspaces of (m) to (c_0), culminating in Theorem 5. Using the result of Phillips mentioned above, we now show that there is no projection defined on (m) to (c_0).[9] This result is perhaps rather surprising, since in all of our previous instances, not only did projections exist, but also there were projections of norm exactly 2. (One might expect to be able to find easily a succession of larger and larger subspaces containing (c_0), for which the greatest lower bounds of the norms of projections would increase unboundedly.) Theorem 5 shows that the nonexistence of a projection on (m) to (c_0) is due in an essential way to the *inseparability* of (m).

Phillips' result is that there is no projection on (m) to (c). It follows easily from this that there is no projection on (m) to (c_0), in view of the following simple theorem.

THEOREM 6. *Suppose that l_1, l_2 are any pair of complementary subspaces in a Banach space L, and that l_2', l_2'' are any pair of complementary subspaces in l_2. Let $L_1 = (l_1 + l_2')$. Then L_1 is closed in L, and L_1 and $L_2 = l_2''$ are complementary subspaces in L.*

PROOF. Let P be the projection of L on l_1 according to l_2. Suppose that $\{x_n\}$ is any sequence of L_1, such that $x_n \to x$ in L, and let $x_n = x_{n1} + x_{n2}'$, where $x_{n1} \in l_1$, $x_{n2}' \in l_2'$. Then $Px_n = x_{n1} \to Px = x_1 \in l_1$, and $(I-P)x_n = x_{n1} + x_{n2}' - x_{n1} \to (I-P)x = x_2' \in l_2'$, by the continuity of P and of $I-P$. Therefore $x_n = x_{n1} + x_{n2}' \to x_1 + x_2' = x$, so that $x \in L_1$, and L_1 is closed. Moreover, l_1 and l_2' are complementary subspaces in L_1.

[9] "Projection *of* (m) *on* (c_0)" of course means the same as "projection *on* (m) *to* (c_0)."

By hypothesis, for any $x \in L$, $x = x_1 + x_2 = x_1 + x_2' + x_2''$, where $x_1 \in l_1$, $x_2' \in l_2'$, $x_2'' \in l_2''$, and this decomposition is unique. Therefore the decomposition $x = (x_1 + x_2') + x_2''$ according to L_1 and L_2 also is unique. This verifies Theorem 6.

If there were a projection of (m) on (c_0), there would be a complementary subspace l_2 to $l_1 = (c_0)$ in (m). The element $X_1 = (1, 1, \cdots) \in (m)$ would have a decomposition $X_1 = x^{(0)} + X$, $x^{(0)} \in (c_0)$, $X \in l_2$. Consider the one-dimensional subspace consisting of all elements of the form tX. By the Hahn-Banach theorem, there exists a projection of norm 1 on any one-dimensional subspace. Thus if $l_2' = \{tX\}$, there exists a complementary subspace l_2'' in l_2. The space $L_1 = (l_1 + l_2')$ of Theorem 6 is (c), and by Theorem 6 and the lemma of Murray already mentioned, there would exist a projection of (m) on (c), contradicting the result of Phillips. Thus there can be no projection of (m) on (c_0).

By an argument identical with that used by Phillips to prove the nonexistence of a projection of (m) on (c), it may also be shown directly that there is no projection of (m) on (c_0). Theorem 6 will be required, however, for another remark to be made below. (It should perhaps be mentioned that (c) is isomorphic to (c_0).[10])

By Theorem 5 and since there is no projection of (m) on (c_0), there is no projection of (m) on any separable subspace $W \supset (c_0)$. (For if Q were a projection of (m) on W, P a projection of W on (c_0), then PQ would be a projection of (m) on (c_0).) The following question now arises: Does there exist a closed linear subspace L of (m), $L \supset (c_0)$, such that there is no projection of (m) on L, and no projection of L on (c_0)? Of course by Theorem 5, such a space L cannot be separable.

An extension theorem found in the paper by Phillips which has been cited[11] is as follows. Let X be any linear subspace of a Banach space Z, and let U be any linear transformation on X to (m). If $y = \{y_i\} = Ux$, the functionals $y_i(x)$ are obviously linear, and by the Hahn-Banach theorem they may be extended to Z with preservation of the norm. Let $\bar{z}_i(z)$ denote the extension of $y_i(x)$. Then $U_1 z = \{\bar{z}_i(z)\}$ is a linear transformation defined on Z with range (m), which coincides with U on X, and $|U_1| = |U|$. As a consequence of this, if X is isomorphic to any subspace (B) of (m), and if there exists any Banach space $W \supset X$ such that there is no projection of W on X, then there is no projection of (m) on (B). For if U is the isomorphism, U_1 its extension on W, and if Q were the projection of (m) on (B), then $U^{-1}QU_1$ would be a projection of W on X.

[10] Banach, op. cit., p. 181.

[11] P. 538, Theorem 7.1, Corollary 7.2.

It follows immediately by the previous paragraph that not only is there no projection of (m) on (c_0) in the usual orientation of (c_0) with respect to (m), but also that there is no projection of (m) on (c_0) in any imbedding whatever of (c_0) in (m). (Let W be (m), let X be (c_0) in the usual imbedding, let (B) be (c_0) in an arbitrary imbedding in (m), and take U to be the identity on (c_0) to (c_0).) Also, since as has been shown by Phillips[12] there exists no projection of (C_1) (the space of functions having only discontinuities of the first kind) on (C), it follows that there is no projection of (m) on (C) in the imbedding of Theorem 4, or in any other imbedding.

Another interesting consequence of Phillips' extension theorem is that for every Banach space $Z \supset (m)$, there is a projection of norm 1 on Z to (m). (Extend to Z the identity transformation of $(m) \subset Z$ into (m).) Also, consider the finite dimensional spaces $l_{\infty,n}$ of sequences $x=(x_1, \cdots, x_n)$, with norm $\|x\| = \max_i |x_i|$. In a similar way, it follows that there is a projection of norm 1 on any Banach space $Z \supset l_{\infty,n}$ to $l_{\infty,n}$. There is probably some connection between this and the result of Theorem 5, since $(c_0) \supset l_{\infty,n}$ for every n. A question which arises here is whether the most general separable Banach space having (c_0) as a subspace is contained in (m) in the same relationship to (c_0) as in Theorem 5. Anyway, if B is any separable Banach space having a subspace equivalent to (c_0), there is a projection of norm 2 of B on (c_0); for let U be the identity on (c_0) in B to (c_0) in (m), extend U to be U_1 on B to (m) by Phillips' theorem, and let the proper range of U_1 be W'. By continuity of U_1, W' is separable. Let P' be a projection of norm 2 on W' to (c_0), as given by Theorem 5. Then $U^{-1}P'U_1$ is the required projection of norm 2 on B to (c_0).

Any separable Banach space Y is equivalent to a subspace (B) of (m).[13] The question of whether there is an infinite dimensional, separable Banach subspace (B) in (m), such that there *is* a projection on (m) to (B), is equivalent to the question of whether there is a separable Banach space Y, such that for every Banach space $W \supset Y$, there is a projection on W to Y. A further question is whether such a subspace is necessarily reflexive. ((c_0) is not reflexive.)

Let Y be any separable Banach space; imbed Y in (C) and (C) in (m). If there is no projection on (C) to Y, then there is no projection on (m) to Y in any imbedding. If there is a projection on (C) to Y, then using Theorem 6 and the facts that any two complementary subspaces of the same closed linear subspace are isomorphic, and that

[12] Loc. cit., p. 539.

[13] A direct imbedding is given by Phillips, loc. cit., p. 524.

there is no projection on (m) to (C), it may be shown that at least either there is no projection on (m) to Y, or else there is no projection on (m) to the complementary subspace of Y in (C). (An illustration of the case where there is no projection on (m) to the complementary subspace in (C) is provided by the case of a finite dimensional $Y \subset (C)$.)

In a paper in preparation on the extension of linear transformations, the writer intends to discuss the questions indicated above, and related questions.

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SEQUENCES OF STIELTJES INTEGRALS[1]

H. M. SCHWARTZ

Statement of results. Sequences of Riemann-Stieltjes integrals[2] have as yet been little studied, only the following fundamental results being known.

THEOREM A (Helly [2]). *Let $g_n(x)$ $(n=1, 2, \cdots)$ be an infinite sequence of real functions defined in the finite closed interval $I=(a, b)$ which satisfy the following two conditions:*

(1) *Total variation of g_n in $I \equiv V_I(g_n) \leqq M$, M a fixed constant,*

(2) $$g_n \to g \text{ on } I, \qquad n \to \infty;$$

then for any function $f(x)$ continuous in I, we have[3]

(3) $$\int f dg_n \to \int f dg.$$

THEOREM B (Shohat [3]). *Let $\{g_n\}$ be a sequence of functions monotonic and uniformly bounded in I and such that*

(4) $g_n \to g$ *on E, E a set dense on I and including the end points a, b of I,*

where g is a monotonic function (all the functions g_n, g monotonic in the same sense); then we have (3) *for any function $f(x)$ for which*

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[1] Presented to the Society, January 1, 1941.

[2] A discussion of such integrals with references is to be found in [1]. (Numbers in brackets refer to the bibliography.)

[3] When the limits of integration are omitted, it is to be understood that they are the end points a, b of I.

(5) $\int f dg_n$ *exist,* $n = 1, 2, \cdots,$

(6) $\int f dg$ *exists.*

Theorem A was rediscovered by Bray [4][4] and a condition weaker than (4), which is both necessary and sufficient, was found by Hildebrandt [5].[5]

The present paper is devoted to a preliminary study of conditions governing convergence of the sequences in question when neither continuity of f nor monotonicity of g_n is assumed. The goal, not attained here, is to find necessary and sufficient conditions to be imposed on the sequence $\{g_n\}$, where the functions g_n belong to the class V of functions of bounded variation in I, in order to insure the validity of (3) for every bounded f for which (5), or (5) and (6) hold true. Two necessary conditions can be given at once. Condition (1) is necessary since it is already necessary if we require that f belong only to the class of functions continuous in I [5]. It is interesting to note, however, that condition (1) is not necessary if we require that f belong only to V, as is seen by the following theorem:

THEOREM 1. *If we have* (2), *then in order that we have* (3) *for every function* f *of* V *for which* (5) *and* (6) *are true, it is sufficient that the sequence* $\{g_n\}$ *be uniformly bounded.*

The other necessary condition is obtained by taking for f functions which are constant everywhere in I except for a single point. It is easily found that the set C of points of convergence of g_n to g must include all the common continuity points of g_n, g as well as the end points a, b of I. Without going into the question of necessity[6] we shall assume that C contains all the continuity points of g (whether they are also continuity points of all g_n or not), that is, that

(7) $g_n \to g$ *on* C, g *in* V, C *consisting of the continuity points of* g *and of* a *and* b.

[4] In Bray's formulation of the theorem condition (2) is replaced by (4); but it is then necessary to specify that g is a function of bounded variation; and by the lemma given below, it is seen that the two formulations are compietely equivalent.

[5] The following additional reference was brought to the author's attention by the referee: Hildebrandt, *Stieltjes integration of the Riemann type*, American Mathematical Monthly, vol. 45 (1938), pp. 265–278, Theorems 2.41; 5.41.

[6] It is to be noted that in Theorem B as well as in its generalization, Theorem 3, conditions (4) and (7) are equivalent, as can be readily proved.

The question arises whether condition (7) represents in the problem under consideration an essential weakening of condition (2). That this is not the case is shown by the following lemma.

LEMMA. *Under the conditions* (1), (5), *and* (6), *if* (3) *is implied by* (2), *it will also be implied by* (7). *If we are considering only a certain f satisfying our conditions, then C need consist of only those continuity points of g which are also discontinuity points of f (in addition to being dense on I and containing the points a and b).*

It is thus no loss of generality to assume condition (2) rather than (7) in the following theorems. When this is the only condition imposed on the sequence $\{g_n\}$, only the following result (besides that given in Theorem 1) has so far been obtained.

THEOREM 2. *If f has at most singularities of the first kind, then* (1), (2), (5), *and* (6) *imply* (3).

A direct generalization of Theorem B is given in the following theorem.

THEOREM 3. *If in addition to* (2) *we assume that*

$$V_I(g_n) \to V_I(g), \qquad g_n, g \text{ in } \mathbf{V}, \tag{8}$$

then (5) *and* (6) *imply* (3). (*Condition* (1) *is of course implied by* (8).)

The next theorem gives a result of wider scope.

THEOREM 4. *To insure the validity of* (3) *it is sufficient to assume in addition to* (1), (2), *and* (5) *that*[7]

(9) *for every limit function* $v(x)$ *of the sequence* $\{V_a^x(g_n)\}$ $\int f dv$ *exists.*

In some applications the interval of integration is infinite and it is therefore of interest to consider that case. This is treated here only briefly, the following results being immediate.[8]

THEOREM 5. *For* $I=(a, b)=(a, \infty)$, *if we assume* (2), (5), *and*

$$\int_a^c f dg_n \to \int_a^c f dg \text{ for every finite } c > a; \qquad \int_a^c f dg \text{ exists,} \tag{10}$$

[7] The existence of at least one limit function of the sequence given in (9) is insured by Helly's theorem of choice [2]. Note also that (6) is implied by (9), as can be seen, for example, by using the integrability condition on p. 542 [1] along with Theorem 2 in [7].

[8] In order to simplify the formulas, we take only one of the limits as infinite.

then a necessary and sufficient condition for the validity of (6) *and* (3) *is that we have either one of the following two conditions* (*e being an arbitrarily assigned positive number*):

$$\left| \int_c^\infty f dg_n \right| < e, \qquad c \geqq c_e, n \geqq N(e, c), \tag{11a}$$

$$\left| \int_a^c f d(g_n - g) \right| < e, \qquad n \geqq n_e, c \geqq C(e, n). \tag{11b}$$

NOTE 1. Condition (10) of the theorem can of course be replaced by the appropriate conditions in the preceding theorems which insure the validity of (10); thus if f is continuous in I, we can replace (10) by the requirement:

$$V_a^c(g_n) \leqq M(c). \tag{12}$$

NOTE 2. If f is bounded in I, then (11a) can be replaced by the relation $V_c^\infty(g_n) < e$, $c \geqq c_e$, $n \geqq N(e, c)$.

NOTE 3. If f is bounded in I and g is of bounded variation for sufficiently large x, then we need assume only the validity of a relation like (4), and (11) can be replaced by the following corresponding relations:

$$\left| \int_{a_k}^\infty f dg_n \right| < e, \qquad k \geqq k_e, n \geqq N(e, k),$$

$$\left| \int_a^{a_k} f d(g_n - g) \right| < e, \qquad n \geqq n_e, k \geqq K(e, n),$$

where $\{a_k\}$ is some sequence in E converging to ∞.

NOTE 4. If we have (12) and if

$$\lim_{c \to \infty} \overline{B}_{x \geqq c} \{ |f(x)| \} \cdot M(c) = 0,$$

then we certainly have (11a). (The above relation will hold for instance when $\lim_{x \to \infty} f(x) = 0$ and $M(c)$ is independent of c.)

By the use of Theorem 5 we can give a direct proof of the existence of a solution of the moments-problem of Stieltjes-Hamburger.[9] This proof as well as other applications of the above theorems will be given elsewhere.

Proof of the lemma. The proof will be based on the following auxiliary lemma.

[9] The proof given in the literature is indirect, involving complex function theory. A discussion of this problem and references are to be found in [6].

LEMMA. *Let g and h be two functions of V which coincide in value on a set D, and let f be a function bounded in I. Then the existence of the integral of f with respect to one of these functions, say g, will imply its existence with respect to the other function h, and the equality of the values of the two integrals, provided D is dense in I and includes the end points of I as well as all those discontinuity points of f at which h has an external saltus.*

PROOF. That $\int f dg = \int f dh$ when both integrals exist follows directly by the definition of Stieltjes integrals and the fact that D is dense in I and includes the end points of I. To prove that $\int f dh$ exists it is clearly sufficient to prove that $\int f dk$ exists, where $k = g - h$; and for that it is only necessary to show that the variation of k over the set of discontinuities of f is zero [1]. This can be readily shown when we note that by our assumption $k(x-0) = k(x+0) = 0$, and that therefore, if we denote the discontinuity points of k by u_i $(i = 1, 2, \cdots)$, the total variation of k over the set obtained by excluding from I arbitrarily small intervals about the points $u_1, \cdots, u_m$, can be made as small as we please provided m is taken sufficiently great; and that further by our assumptions, f is continuous at each u_i.

To prove the lemma, we note that by (1) the sequence $\{\int f dg_n\} \equiv \{J_n\}$ is bounded and it therefore contains at least one convergent subsequence. Let $\{J_{n'}\}$ be any such convergent subsequence and denote its limit by J; let $\{g_{n''}\}$ be a convergent subsequence of the function-sequence $\{g_{n'}\}$ (existing by Helly's theorem of choice [2]), and denote its limit function by g^*. Now it can easily be proved that by the condition given in the lemma, no limit function of $\{g_n\}$ can have an external saltus at a discontinuity point of f. We can apply therefore the above auxiliary lemma and conclude that $\int f dg^*$ exists and equals $\int f dg$. Hence by our assumption, it follows that $\int f dg_{n''} \to \int f dg$, and consequently also $J = \int f dg$. But $\{J_{n'}\}$ was an arbitrary convergent subsequence of $\{J_n\}$; hence $\{J_n\}$ has the unique limit J, that is, we have (3).

Proof of Theorem 1. It is clearly sufficient to consider f monotonic. By the formula of integration by parts for Stieltjes integrals [1], it is seen that the theorem will follow if we show that

$$(*) \qquad \int g_n df \to \int g df.$$

To this end, we employ Lebesgue's theorem on the convergence of sequences of Lebesgue integrals with boundedly convergent integrands.

Putting $A=f(a)$, $B=f(b)$, $G_n(y)\equiv g_n(f^{-1}(y))$, $G(y)\equiv g(f^{-1}(y))$, we have [1]:

$$\int g_n df = L\int_A^B G_n(y)dy, \qquad \int g df = L\int_A^B G(y)dy;$$

and as by our assumptions, all the conditions of Lebesgue's theorem are satisfied, it follows that $\int G_n(y)dy \to \int G(y)dy$, and that we have (*).

Proof of Theorem 2. This theorem follows from the following three lemmas (Lemma 2 is more general than the present use of it requires).

LEMMA 1. *Theorem 2 is true if f is in V.*

PROOF. Let $f=f_c-f_d$ be the decomposition of f into its continuous part f_c and its discontinuous part f_d [1]. By Theorem A, $\int f_c dg_n \to \int f_c dg$, and it remains to be shown that we also have the relation $\int f_d dg_n \to \int f_d dg$, or the relation corresponding to (*) above. Now $\int g_n df_d = \sum_m d_m g_n(u_m)$, where $d_m = f(u_m+0)-f(u_m-0)$, and as the series $\sum d_m g_n(u_m)$ converges uniformly with respect to n, being dominated by

$$\overline{B}\{|g_n|\}\cdot\sum|d_m|$$

which is finite by our assumptions, it follows that

$$\lim_{n\to\infty}\int g_n df_d = \sum_m d_m g(u_m) = \int g df_d.$$

LEMMA 2. *If the sequence of bounded functions f_m converges uniformly in I to the bounded function f, and if we have* (5), (6), *and* (3) *with f replaced by f_m $(m=1, 2, \cdots)$ then we have* (5), (6), *and* (3), *provided we assume* (1) *and the finiteness of $V_I(g)$.*

PROOF. We show first that the existence of $\int f_m dg$ and the uniform convergence of f_m to f in I implies the relation

$$(*) \qquad \int f dg \text{ exists}, \qquad \int f_m dg \to \int f dg,$$

g being assumed to belong to V. Let

$$s_{mk} = \sum_{i=1}^{r_k} f_m(u_i)[g(x_i) - g(x_{i-1})]$$

be the kth Riemann sum of a sequence whose limit for $k\to\infty$ is $\int f_m dg$, and let s_k represent the corresponding sum for f. As

$$|s_{mk} - s_k| \leqq \overline{B}_{x \text{ in } I}\{|f_m - f|\}\cdot V_I(g),$$

we see that s_{mk} approaches its limit s_k uniformly with respect to k, and that therefore $\lim_{k\to\infty} s_k$ and $\lim_{m\to\infty} \int f_m dg$ exist and are equal. Relation (*) now follows from the fact that the first limit is independent of the choice of the division points x_i and the between-points u_i in the Riemann sums.

Now by (1),

$$\left| \int f dg_n - \int f_m dg_n \right| \leqq \overline{B}_{x \text{ in } I} \{ | f - f_m | \} \cdot M,$$

that is, $\int f_m dg_n$ converges to $\int f dg_n$ uniformly with respect to n. Hence

$$\lim_n \int f dg_n = \lim_n \lim_m \int f_m dg_n = \lim_m \lim_n \int f_m dg_n$$

$$= \lim_m \int f_m dg = \int f dg, \qquad \text{by (*).}$$

LEMMA 3. *If f is a function having at most singularities of the first kind in I, then there exists a sequence $\{f_m\}$ converging to f uniformly in I, where each f_m is in V and is continuous at the continuity points of f.*

PROOF. We have to show that we can assign to every given positive and arbitrarily small number e, a function f_e, which is in V, is continuous at every continuity point of f, and approximates f within e. By assumption, $f(x-0)$ and $f(x+0)$ exist, and we can therefore assign to every x of the semi-open interval $(a, b]$,[10] a left-side interval (x', x), and to every x of $[a, b)$, a right-side interval (x, x''), in each of which the oscillation of f is less than $e/4$. By the Heine-Borel covering theorem, there exists a finite set of points x_i in I such that the sum of the corresponding intervals $(x_i', x_i'') \equiv I_i$ (including the intervals $[a, a'')$ and $(b', b]$) covers I, and we may assume without loss of generality that the overlapping parts (x_{i+1}', x_i'') of consecutive intervals I_i (necessarily nonvacuous, as the I_i are open) lie inside (x_i, x_{i+1}). It can now be seen without difficulty that we can take as the required function f_e, a function defined as follows: It has the values $f(x_i)$, $f(x_i-0)$, and $f(x_i+0)$, respectively for $x=x_i$, for x in (x_{i-1}'', x_i), and for x in (x_i, x_{i+1}'); and it is linear in the remaining intervals.

It is apparent how Theorem 2 can be derived by the use of the above three lemmas. We need only observe that the functions f_m of Lemma 3 (with f of Theorem 2) satisfy the relations (5) and (6) (with $f=f_m$) because they are in V, and have no common singularity

[10] The *parenthesis* will represent the *open*, and the *bracket*, the *closed* end of an interval.

points with the functions g_n, g (by their definition, in view of (5) and (6)). It is in fact known that two functions of bounded variation which have no common discontinuity points, are integrable with respect to each other.[11]

Proof of Theorem 3. Conditions (2) and (8) are equivalent [7] to (2) and the relation

$$(8') \qquad V_a^x(g_n) \to V_a^x(g), \qquad n \to \infty\,;\ x \text{ in } I.$$

Hence, denoting by $g_n = p_n - q_n$, $g = p - q$ the Jordan decompositions of the functions g_n, g, we readily see that all the conditions of Theorem B apply to the two sequences $\{p_n\}$, $\{q_n\}$ ($\int f dp_n$, $\int f dq_n$ exist because $\int f dg_n$ exist [1]), and that therefore: $\int f dp_n \to \int f dp$, $\int f dq_n \to \int f dq$, and hence (3) is satisfied.

The theorem can also be proved directly as follows:[12] Let

$$s_{nk} = \sum_{i=1}^{r_k} f(u_{i,k})\left[g_n(x_{i,k}) - g_n(x_{i-1,k})\right]$$

represent the kth Riemann sum of a sequence of such sums which converges to $\int f dg_n \equiv J_n$ when $k \to \infty$, then:

$$(*) \qquad |J_n - s_{nk}| \leqq \sum_{i=1}^{r_k} \mathrm{Osc}_i f(x) V_i(g_n),$$

the subscript i referring to the ith subinterval $(x_{i-1,k}, x_{i,k})$. By (6), given a positive number h, there exists a number $K = K(h)$ such that $O_k \equiv \sum_i \mathrm{Osc}_i f(x) V_i(g) < h$, $k \geqq K(h)$. Now let e be a preassigned positive number. Fixing first $k' \geqq K(e/2)$, we can then choose, in view of (8'), a number $N = N(e, k')$ so that for all $n \geqq N$, we have

$$|V_a^{x_{i,k'}}(g_n) - V_a^{x_{i,k'}}(g)| < \frac{e}{4r_{k'}U} \equiv e', \quad U = \mathrm{Osc}_I f;\ i = 0, 1, \cdots, r_{k'}.$$

Then

$$\sum_{i=1}^{r_{k'}} \mathrm{Osc}_i f(x) V_i(g_n) < O_{k'} + 2e' r_{k'} U < e/2 + e/2 = e,$$

$$k' \geqq K(e/2) = k_e;\ n \geqq N(e, k').$$

[11] We note, incidentally, that by Lemma 3 and the first part of the proof of Lemma 2, it follows that *a bounded function which has at most singularities of the first kind is integrable with respect to every function of* $\mathbf{V}$ *with which it has no common singularities.*

[12] This proof, which is different from that given in [3] for Theorem B, was found by the author in the winter of 1939, when most of the results given in this paper were developed.

Combining this with (*), and remembering that e was arbitrarily assigned, we finally conclude that $\lim_{n\to\infty} J_n = \lim_k \lim_n s_{nk} = \int f dg$, that is, that we have (3).

Proof of Theorem 4. Let us suppose first that the limit function $v(x)$ occurring in (9) is unique. Then if $g_n = p_n - q_n$ represent the Jordan decompositions of the functions g_n, we find:

$$2p_n(x) \to v(x) + g(x) + g(a) \equiv p^*(x), 2q_n \to v(x) - g(x) + g(a) \equiv q^*(x).$$

Since by (6) and (9), the integrals of f with respect to p^* and q^* exist, we can now proceed exactly as in the first proof of Theorem 3.

In the general case, let $\{J_{n'}\}$ be an arbitrary convergent subsequence of $\{J_n\} \equiv \{\int f dg_n\}$. By Helly's theorem of choice, in view of (1) and (2), there exists a convergent subsequence $\{v_{n''}(x)\}$ of the sequence $\{v_{n'}(x)\} \equiv \{V_a^x(g_{n'})\}$, and it follows by the preceding discussion that $J_{n''} \to \int f dg$. By a familiar reasoning it then follows that we also have the relation $J_n \to \int f dg$, that is, (3).

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HEAT CONDUCTION IN AN INFINITE COMPOSITE SOLID[1]

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1. **Introduction.** The problem of one-dimensional heat conduction in finite or semi-infinite composite solids has been investigated extensively. The doubly-infinite case, however, seems to have been treated only for special initial temperature distribution functions.[2]

The purpose of the present paper is to treat the general case. The Laplace transformation is used formally in §2 to discover the solution, which is then rigorously established in §3. In §4, finally, a uniqueness theorem is proved, under more restrictive conditions on the initial distribution function.

Lebesgue integrals are used throughout.

2. **The formal solution.** Consider two plane-boundary semi-infinite homogeneous solids, composed of different materials, placed in perfect thermal contact. If the conduction of heat takes place in only one dimension, perpendicular to the interface, the temperature $U(x, t)$ satisfies the following differential system:

$$\frac{\partial U}{\partial t} = a_\nu \frac{\partial^2 U}{\partial x^2}; \qquad t > 0;\ x < 0,\ \nu = 1;\ x > 0,\ \nu = 2; \tag{1}$$

$$\lim_{t \to 0} U(x, t) = f(x); \qquad x \neq 0; \tag{2}$$

$$\lim_{x \to -0} U(x, t) = \lim_{x \to +0} U(x, t); \qquad t > 0; \tag{3}$$

$$\lim_{x \to -0} k_1 \frac{\partial U}{\partial x} = \lim_{x \to +0} k_2 \frac{\partial U}{\partial x}; \qquad t > 0; \tag{4}$$

where x is the perpendicular distance from the interface, t is time, a_ν and k_ν are the thermal diffusivities and conductivities, respectively, of the two materials and are positive constants, and $f(x)$ is a known function, defined for all real x except $x = 0$, whose properties will be specified later.

Denoting the common limit in equation (3) by $\Phi(t)$, the solution can be written in the well known form[3]

[1] Presented to the Society, April 5, 1941.

[2] Cf. Riemann-Weber, *Die partiellen Differential-Gleichungen der mathematischen Physik*, 5th edition, 1912, vol. 2, p. 98.

[3] Cf. H. S. Carslaw, *The Mathematical Theory of the Conduction of Heat in Solids*, 2d edition, London, 1921, §§18, 23.

(5)[4] $$U(x,t) = V_\nu(x,t) + \frac{|x|}{2(\pi a_\nu)^{1/2}} \int_0^t \Phi(t-\xi)\xi^{-3/2} \exp\left[-\frac{x^2}{4a_\nu \xi}\right] d\xi,$$

where

$$V_1(x,t) = \frac{1}{2(\pi a_1 t)^{1/2}} \int_{-\infty}^0 f(\xi) \left\{ \exp\left[-\frac{(x-\xi)^2}{4a_1 t}\right] - \exp\left[-\frac{(x+\xi)^2}{4a_1 t}\right] \right\} d\xi,$$

and $V_2(x, t)$ is obtained from $V_1(x, t)$ on replacing a_1 by a_2 and integrating from 0 to $+\infty$.

Now apply the Laplace transformation to equations (4) and (5), denoting the transforms of U, V, and so on, by u, v, and so on, respectively:

(4′) $$\lim_{x\to -0} k_1 \frac{\partial u(x,s)}{\partial x} = \lim_{x\to +0} k_2 \frac{\partial u(x,s)}{\partial x},$$

(5′)[5] $$u(x,s) = \phi(s) \exp\left[-|x|(s/a_\nu)^{1/2}\right] + v_\nu(x,s).$$

The unknown function $\phi(s)$ can be eliminated between equations (4′) and (5′), and the solution is then obtained by applying the inverse Laplace transformation to $u(x, s)$:

$$U(x,t) = V_1(x,t) + \frac{k_1}{A a_1 (\pi t)^{1/2}} \int_{-\infty}^0 f(\xi) \exp\left[-\frac{(x+\xi)^2}{4a_1 t}\right] d\xi + \frac{k_2}{A a_2 (\pi t)^{1/2}} \int_0^\infty f(\xi) \exp\left[-\frac{(x a_2^{1/2} - \xi a_1^{1/2})^2}{4a_1 a_2 t}\right] d\xi; \quad x<0,\ t>0;$$

(6)
$$U(x,t) = V_2(x,t) + \frac{k_2}{A a_2 (\pi t)^{1/2}} \int_0^\infty f(\xi) \exp\left[-\frac{(x+\xi)^2}{4a_2 t}\right] d\xi + \frac{k_1}{A a_1 (\pi t)^{1/2}} \int_{-\infty}^0 f(\xi) \exp\left[-\frac{(x a_1^{1/2} - \xi a_2^{1/2})^2}{4a_1 a_2 t}\right] d\xi; \quad x>0,\ t>0,$$

where $A = (k_1(a_2)^{1/2} + k_2(a_1)^{1/2})/(a_1 a_2)^{1/2}$.

[4] Throughout, $x<0$ when $\nu=1$, $x>0$ when $\nu=2$.

[5] The first term in the right member is obtained by the "Faltung" rule. For it and the specific transformations used, cf. G. Doetsch, *Theorie und Anwendung der Laplace Transformation*, Berlin, 1937, particularly the table of transformations in Appendix 2.

3. **The solution established.** We have the following theorem.

THEOREM 1. *If $f(x)$ is Lebesgue integrable over any finite interval, and if there exist two positive numbers B, b, such that $|f(x)| < B \exp |bx|$, then the function $U(x, t)$ defined by equations* (6) *is a solution of the boundary value problem* (1), (3), (4). *The initial condition* (2) *is satisfied uniformly in x in any finite interval $0 < x_1 \leqq |x| \leqq x_2$ in which $f(x)$ is continuous.*

In the classical proofs of similar theorems[6] $f(x)$ is always a bounded function. In adapting such methods of proof to the present situation it is only necessary to investigate the effect of the unboundedness of $f(x)$. A complete proof is given here for only one typical integral in each case. The right members of (6) obviously converge.

(a) *The differential equation.* If $U(x, t)$ is differentiated with respect to t under the integral sign, one of the resulting integrals is of the form

$$J(x, t) \equiv t^{-5/2} \int_0^\infty (x \pm \xi)^2 f(\xi) \exp\left[-\frac{(x \pm \xi)^2}{4t} \right] d\xi.$$

We prove that, if $x > 0$, $0 < t_1 \leqq t \leqq t_2$, $J(x, t)$ converges uniformly in t. From the hypotheses on $f(x)$,

$$|J(x, t)| \leqq B t_1^{-5/2} \int_0^\infty (x \pm \xi)^2 e^{b\xi} \exp\left[-\frac{(x \pm \xi)^2}{4t_2} \right] d\xi.$$

Since the right member converges, and does not contain t, $J(x, t)$ converges uniformly in t.

(b) *The initial condition.* The solutions (6) are composed of two essentially distinct types of integrals, according as x and ξ have the same or opposite signs. We first prove that, in the first case, the integral vanishes with t.

Consider

$$J(x, t) \equiv \frac{1}{2(\pi t)^{1/2}} \int_0^\infty f(\xi) \exp\left[-\frac{(x + \xi)^2}{4t} \right] d\xi, \qquad x > 0.$$

Make the change of variable $\xi = -x + 2\zeta t^{1/2}$, then:

$$J(x, t) = \pi^{-1/2} \int_{x/2t^{1/2}}^\infty f(-x + 2\zeta t^{1/2}) e^{-\zeta^2} d\zeta.$$

From the hypotheses on $f(x)$,

[6] Cf. Carslaw, loc. cit., p. 31; Goursat, *Cours d'Analyse Mathématique*, 4th edition, Paris, 1927, vol. 3, chap. 29.

$$|J(x, t)| \leqq Be^{-bx_1}\pi^{-1/2}\int_{x/2t^{1/2}}^{\infty} \exp\left[-\zeta(\zeta - 2bt^{1/2})\right]d\zeta.$$

If $x \geqq x_1 > 0$, and $t < x_1/8b$, then

$$|J(x, t)| \leqq \frac{4B}{x_1}(t/\pi)^{1/2} \exp\left[-bx_1 - x_1^2/8t\right].$$

Hence $J(x, t)$ approaches zero with t, uniformly in x in any semi-infinite interval $0 < x_1 \leqq x$.

An inspection of (6) now shows that, in order to prove that the initial condition (2) is satisfied it is sufficient to prove that

$$\lim_{t\to 0}\left[J_1(x, t) - f(x)\right] = 0$$

where

$$J_1(x, t) \equiv \frac{1}{2(\pi t)^{1/2}}\int_0^{\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4t}\right]d\xi, \qquad x > 0,$$

the approach to the limit being uniform in x in any interval $0 < x_1 \leqq x \leqq x_2$. Given any $\epsilon > 0$, we can choose a $\delta > 0$, independent of x and t, such that $|f(\xi) - f(x)| < \epsilon$ if $|x - \xi| < \delta$, $\delta < x_1$. Having chosen δ, we can write

$$(7) \quad |J_1(x, t) - f(x)| \leqq \int_0^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\infty} + \frac{|f(x)|}{\pi^{1/2}}\int_{x/2t^{1/2}}^{\infty} e^{-\xi^2}d\xi,$$

where in each of the first three integrals the integrand is

$$\frac{1}{2(\pi t)^{1/2}}|f(\xi) - f(x)| \exp\left[-\frac{(x-\xi)^2}{4t}\right]d\xi.$$

Denote the right member of (7) by $I_1 + I_2 + I_3 + I_4$. The unboundedness of $f(x)$ is important in I_1, I_3, and I_4. I_2 is easily seen to be less than ϵ. In I_1 make the change of variable $\xi = x - 2\zeta t^{1/2}$. Then

$$I_1 + I_4 \leqq 3Be^{bx_1}\pi^{-1/2}\int_{\delta/2t^{1/2}}^{\infty} e^{-\zeta^2}d\zeta,$$

which obviously approaches zero with t, uniformly in x.

From the hypotheses on $f(x)$,

$$I_3 \leqq \frac{B}{(\pi t)^{1/2}}\int_{x+\delta}^{\infty} \exp\left[b\xi - \frac{(x-\xi)^2}{4t}\right]d\xi.$$

Make the change of variable $\xi=x+8t\zeta/\delta$; if $t<\delta/8b$,

$$I_3 \leqq \frac{8B}{\delta} e^{bx_2}(t/\pi)^{1/2} \int_{\delta/8t}^{\infty} e^{-\zeta} d\zeta,$$

which obviously approaches zero with t, uniformly in x.

(c) *The interface conditions.* Equations (3) and (4) will obviously be satisfied if x can approach zero within the integral signs. The proof is given for one typical case:

$$J(x, t) \equiv t^{-3/2} \int_0^{\infty} (x \pm \xi) f(\xi) \exp\left[-\frac{(x \pm \xi)^2}{4t}\right] d\xi.$$

If $0<t_1 \leqq t \leqq t_2$, $x>0$,

$$\begin{aligned} |J(x, t) - J(0, t)| &\leqq x t^{-3/2} \int_0^{\infty} |f(\xi)| \exp\left[-\frac{(x-\xi)^2}{4t}\right] d\xi \\ &+ t^{-3/2} \int_0^{\infty} \xi |f(\xi)| \exp\left[\frac{-\xi^2}{4t}\right] \left|1 - \exp\left[-\frac{x^2 - 2x\xi}{4t}\right]\right| d\xi. \end{aligned} \tag{8}$$

Denote the right member of (8) by I_1+I_2. In I_1 make the change of variable $\xi=x+2\zeta t^{1/2}$:

$$I_1 \leqq \frac{2Bx}{t} e^{bx} \int_{-x/2t^{1/2}}^{\infty} \exp\left[-\zeta(\zeta - 2bt^{1/2})\right] d\zeta.$$

If the range of integration is split at $\zeta=1+2bt^{1/2}$, it is easily seen that

$$I_1 \leqq \frac{2Bx}{t_1} e^{bx}\left\{1 + \pi^{1/2} \exp\left[2b(1 + 2bt_2^{1/2})t_2^{1/2}\right]\right\},$$

which approaches zero with x, uniformly in t.

Since $|1-\exp y| \leqq |y| \exp |y|$, we have

$$I_2 \leqq Bxt^{-3/2} \int_0^{\infty} \xi \left|\frac{x - 2\xi}{4t}\right| \exp\left[b\xi - \frac{\xi^2}{4t} + \left|\frac{x^2 - 2x\xi}{4t}\right|\right] d\xi.$$

Split the range of integration at $\xi=3+4bt$; if $x<1$,

$$\begin{aligned} I_2 \leqq Bxt_1^{-5/2} \exp\left[\frac{x^2}{4t_1}\right] \Bigg\{ &\int_0^{3+4bt_2} \xi(1 + 2\xi) \exp\left[b\xi + \frac{\xi}{2t_1}\right] d\xi \\ &+ \int_{3+4bt_1}^{\infty} \xi(1 + 2\xi) \exp\left[-\xi/4t_2\right] d\xi \Bigg\}, \end{aligned}$$

which approaches zero with x, uniformly in t.

4. Uniqueness. We first state the following definition.

DEFINITION. *A function $U(x, t)$ defined for all real x and positive t will be said to satisfy conditions* A *if*:

(A_1) *$U(x, t)$ is a continuous function of x in $-\infty < x < \infty$ for any fixed $t > 0$.*

(A_2) *$U(x, t)$ is a continuous function of t in any finite interval $0 < t_1 \leqq t \leqq t_2$, the continuity being uniform with respect to x in any finite interval $x_1 \leqq x \leqq x_2$.*

(A_3) *$U(x, t)$ is bounded for all real x, all positive t.*

(A_4) *$\partial U(x, t)/\partial t$ exists and satisfies (A_1) and (A_2).*

(A_5) *$\partial U(x, t)/\partial x$ and $\partial^2 U(x, t)/\partial x^2$ exist and are continuous functions of x in $-\infty < x < 0$ and $0 < x < \infty$, for any fixed $t > 0$.*

(A_6) *$|\partial U(x, t)/\partial x| < M/t^{1/2}$, $t > 0$, $x \neq 0$; M is a constant independent of t and x.*

(A_7) *$U(x, t) \to 0$ as $|x| \to \infty$, uniformly in t in any finite interval $0 < t \leqq t_1$.*

(A_8) *$U(x, t) \to 0$ as $t \to 0$, uniformly in x in any finite interval $0 < x_1 \leqq |x| \leqq x_2$.*

THEOREM 2. *The function $U(x, t) \equiv 0$ is the only function satisfying conditions* A *that is a solution of the boundary value problem* (1)–(4), *with $f(x) \equiv 0$.*

PROOF. Consider

$$J(x, t) \equiv \frac{1}{2} \int_{-x}^{x} \frac{k_\nu}{a_\nu} [U(\xi, t)]^2 d\xi, \qquad x > 0, t > 0.$$

Assume that, for some $x_1 \neq 0$, $t_1 \neq 0$, $|U(x_1, t_1)| = C > 0$. From condition (A_1), $|U(x, t_1)| > C/2$ in some η-neighborhood of x_1. Hence $J(x, t_1) > D$, any $x > |x_1| + \eta$, $t_1 > 0$, where

$$D = \frac{k_\nu C^2 \eta}{4a_\nu}; \qquad \eta < |x_1|\,; \ \nu = 1 \text{ if } x_1 < 0; \ \nu = 2 \text{ if } x_1 > 0.$$

If M is the bound of condition (A_6), we can choose a fixed $x_2 > |x_1| + \eta$ such that

$$(9) \qquad \begin{aligned} |U(x_2, t)| &< \frac{D}{4k_2 M t_1^{1/2}}, \qquad 0 < t \leqq t_1, \\ |U(-x_2, t)| &< \frac{D}{4k_1 M t_1^{1/2}}, \qquad 0 < t \leqq t_1. \end{aligned}$$

Having chosen x_2, consider $J(x_2, t)$. From conditions (A_1), (A_2), (A_4),

$J(x_2, t)$ can be differentiated with respect to t under the integral sign; using equation (1),

$$\frac{\partial J(x_2, t)}{\partial t} = \int_{-x_2}^{x_2} k_r U(x, t) \frac{\partial^2 U(x, t)}{\partial x^2} dx.$$

From condition (A_6) this may be integrated by parts; using equations (3) and (4),

$$\frac{\partial J(x_2, t)}{\partial t} = \left[k_r U(x, t) \frac{\partial U(x, t)}{\partial x} \right]_{x=-x_2}^{x=x_2} - \int_{-x_2}^{x_2} k_r \left[\frac{\partial U(x, t)}{\partial x} \right]^2 dx, \qquad 0 < t \leqq t_1.$$

Since the last term is nonpositive, $\partial J/\partial t$ is not greater than the sum of the first two terms. Hence, from (A_6) and equations (9)

$$\frac{\partial J(x_2, t)}{\partial t} < \frac{D}{2(t_1 t)^{1/2}}, \qquad 0 < t \leqq t_1.$$

From conditions (A_3), (A_5), $J(x_2, +0) = 0$. Therefore

$$J(x_2, t_1) < \frac{D}{2t_1^{1/2}} \int_0^{t_1} t^{-1/2} dt = D.$$

But $J(x_2, t_1) > D$. Thus the assumption that $U(x_1, t_1) \neq 0$ leads to a contradiction if $x_1 \neq 0$, $t_1 \neq 0$, and by the continuity conditions (A_1) and (A_5) the theorem is proved.

We are now in a position to prove a more general uniqueness theorem.

DEFINITION. *Let $f(x)$ be defined and continuous for all real x except $x = 0$. A function $U(x, t)$ will be said to satisfy conditions* B *with respect to $f(x)$ if*:

(B_1) *$U(x, t)$ satisfies (A_1)–(A_6) inclusive.*

(B_2) *$U(x, t) \to f(x)$ as $t \to 0$, uniformly in x in any finite interval $0 < x_1 \leqq |x| \leqq x_2$.*

(B_3) *$U(x, t) - f(x) \to 0$ as $|x| \to \infty$, uniformly in t in any finite interval $0 < t \leqq t_1$.*

THEOREM 3. *Let $f(x)$ be defined and continuous for all real x except $x = 0$. Then the boundary value problem (1)–(4) has at most one solution satisfying conditions* B *with respect to $f(x)$.*

PROOF. If there are two solutions, let $U(x, t)$ be their difference. Then $U(x, t)$ satisfies conditions A and hence vanishes identically.

Finally, it remains to determine conditions on $f(x)$ under which the

solution $U(x, t)$ defined by equations (6) will satisfy conditions B with respect to $f(x)$. Sufficient conditions are given by the following theorem.

THEOREM 4. *Let $f(x)$ be continuous except at $x=0$, and bounded for all real x, and let the following limits exist:*

$$f_1 = \lim_{x\to-\infty} f(x), \qquad f_2 = \lim_{x\to+\infty} f(x).$$

Then $U(x, t)$ as defined by equations (6) *satisfies conditions* B *with respect to $f(x)$.*

PROOF. Conditions (A_1)–(A_5) and (B_2) are easily seen to be satisfied. We give the proof for (A_6) and (B_3). An inspection of (6) shows that it is sufficient to prove (A_6) for integrals of the form

$$J(x, t) \equiv \frac{1}{2(\pi t)^{1/2}} \int_0^\infty f(\xi) \exp\left[-\frac{(x \pm \xi)^2}{4t}\right] d\xi, \qquad x > 0, t > 0.$$

By differentiating inside the integral sign and splitting the range of integration at $x=\xi$ it is easily found that

$$\left|\frac{\partial J(x, t)}{\partial x}\right| < \frac{N}{(\pi t)^{1/2}},$$

where N is the upper bound of $f(x)$.

For (B_3) it is sufficient to prove that

$$J_1(x, t) \to 0 \qquad \text{as } x \to +\infty, \tag{10}$$

$$J_2(x, t) - f(x) \to 0 \qquad \text{as } x \to +\infty, \tag{11}$$

the approach being uniform in t in $0 < t \leqq t_1$, where

$$J_1(x, t) \equiv t^{-1/2} \int_0^\infty f(\xi) \exp\left[-\frac{(x+\xi)^2}{4t}\right] d\xi, \qquad x > 0, t > 0,$$

$$J_2(x, t) \equiv \frac{1}{2(\pi t)^{1/2}} \int_0^\infty f(\xi) \exp\left[-\frac{(x-\xi)^2}{4t}\right] d\xi, \qquad x > 0, t > 0.$$

To prove (10) make the substitution $\xi = -x + 2\zeta t^{1/2}$ in J_1:

$$|J_1(x, t)| \leqq 2N \int_{x/2t^{1/2}}^\infty e^{-\zeta^2} d\zeta < \frac{4Nt_1^{1/2}}{x} \exp\left[-\frac{x^2}{4t_1}\right],$$

which clearly approaches zero as x approaches infinity, uniformly in t in $0 < t \leqq t_1$.

To prove (11), we have

$$J_2(x,t) - f(x) = \frac{1}{2(\pi t)^{1/2}} \int_0^\infty \{f(\xi) - f(x)\} \exp\left[-\frac{(x-\xi)^2}{4t}\right] d\xi + \frac{f(x)}{\pi^{1/2}} \int_{x/2t^{1/2}}^\infty e^{-\xi^2} d\xi, \qquad x > 0,\ t < t \leqq t_1.$$

In the first term of the right member, make the change of variable $\xi = x + 2\zeta t^{1/2}$. Then

$$|J_2(x,t) - f(x)| \leqq \frac{3N}{\pi^{1/2}} \int_{x/4t^{1/2}}^\infty e^{-\zeta^2} d\zeta + \frac{1}{\pi^{1/2}} \int_{-x/4t^{1/2}}^\infty |f(x) - f(x + 2\zeta t^{1/2})|\, e^{-\zeta^2} d\zeta. \tag{12}$$

Since $f(x)$ approaches a limit as $x \to \infty$, the second term of the right member of (12) can be made less than any $\epsilon > 0$ by choosing $x > X$, independent of t. The first term of the right member of (12) clearly approaches zero as required in order that (11) be true.

It may be remarked that the conditions imposed on $f(x)$ in Theorem 4 are merely sufficient, and not necessary. In particular, $f(x)$ need not have limits as $|x| \to \infty$. For example, $U(x,t) - x$ as defined by (6) satisfies conditions B if $f(x) = x$. However, in most physical applications $f(x)$ will satisfy the hypotheses of this theorem, since the problem considered here is an approximation, for small time values, to the finite composite solid problem.

UNIVERSITY OF CALIFORNIA

FIFTIETH ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF PUBLICATION

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